

NEW FIRST COURSE IN ALGEBRA

HAWKES - LUBY - TOUTON

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The Citizens
by James Thomson
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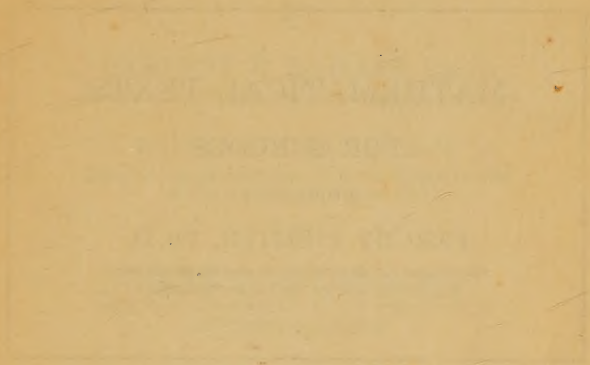
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Cornel Van der Kerk



MATHEMATICAL TEXTS
FOR SCHOOLS

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NEW FIRST COURSE IN ALGEBRA

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PREFACE

In this New First Course in Algebra the authors have endeavored to retain all those features of their revised First Course in Algebra which met widespread approval and to introduce some modifications made necessary by slight changes in requirements for the first year in algebra.

Care has been taken to make such changes in arrangement, in material, and in treatment as experience has shown desirable. Hence increased emphasis has been placed on oral exercises, and the meaning of the formula has been given additional stress. It seems best to reserve for a second course in algebra certain parts of factoring formerly included in first-year work. Wherever possible, the aim has been to make the definitions and the exposition clearer and the illustrative examples simpler and more illuminating. The barest elements of ratio and proportion have been included in the chapter on simple fractional equations.

A new feature is Chapter II on graphs, where the fact is recognized that graphs have an informative use which is widely exemplified in recent periodicals and the daily press. The underlying principles are easy to comprehend, and a simple treatment leads naturally and easily to the mathematical and more scientific use of graphs given in Chapter XIX.

Another feature of this book is the inclusion of numerous lists of miscellaneous review exercises.

In some sections of the country there is a desire to present trigonometry in first-year algebra, and in other

places such work is an actual requirement. This condition has been met by the inclusion of an introduction to trigonometry.

It will be found that the treatment and the material included conform to the recommendations of the various examining bodies, whose reports and revised requirements are of importance.

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NEW FIRST COURSE IN ALGEBRA

CHAPTER I

INTRODUCTION

1. Numbers and symbols in arithmetic. In arithmetic, integers or integral numbers, such as 5, 18, 203, etc., are used to indicate the number of objects or units in a given group which have been or which can be counted. In each case these symbols for integers consist exclusively of the Arabic numerals 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9.

When in the study of arithmetic one's attention is called to less than or more than all the parts of one object, the amount considered is represented by a fractional number, such as $\frac{1}{2}$, $\frac{3}{4}$, $\frac{5}{8}$, 0.8, and 1.25.

The fundamental processes of arithmetic included addition, subtraction, multiplication, and division, which were indicated by the symbols +, −, ×, and ÷ respectively.

The number symbols and the symbols of operation with numbers which were used in arithmetic will be used in algebra as well. One of the striking features which will appear as the student enters upon his work in algebra will be the extent to which additional symbols are used to represent numbers.

2. Number symbols of algebra. In arithmetic, letters are rarely used for numbers, but in algebra, letters as well as Arabic numerals are used to represent numbers.

Thus, in the statement $\text{interest} = \text{principal} \times \text{rate}$, i may be used to represent the number of dollars in the interest paid, p , the number of dollars in the principal or sum borrowed, and r , the rate of interest. Then the fact that $\text{interest} = \text{principal} \times \text{rate}$ is expressed in terms of these letters by

$$i = p \times r$$

The practice of representing numbers by letters illustrates one of the main features of algebra as distinguished from arithmetic. By this means we are frequently able to condense into a compact form a statement which is much longer and often less clear when expressed verbally.

3. Symbols of multiplication in algebra. In addition to the symbol $\times$, which is used in both arithmetic and algebra to indicate the process of multiplication, frequent use is made in algebra of the dot ($\cdot$) placed between two number symbols, as in $a \cdot b$, to indicate the multiplication of a and b . The product of numbers is also indicated by writing the number symbols in immediate succession, as in $2a$, $5x$, cd , etc.

Hence the product of five and t may be expressed as $5 \times t$, as $5 \cdot t$, or as $5t$, of which the last method is the most common in algebra.

A number multiplied by itself is said to be *squared*, and the resulting product is called the *square* of the original number. The symbol for the operation of squaring a number is a small 2 placed at the upper right of the number squared. Thus $2 \text{ squared} = 2 \cdot 2 = 2^2 = 4$.

ORAL EXERCISES

Express the following in words:

- | | |
|-----------------------|-----------------------------------|
| 1. 3×4 . | 14. $p \cdot r = i$. |
| 2. $5 \times a$. | 15. $A = lw$. |
| 3. $2 \cdot s$. | 16. $d = rt$. |
| 4. $7c$. | 17. $p = 2l + 2w$. |
| 5. $s + 5$. | 18. $A = \frac{1}{2} \times ab$. |
| 6. $c - 6$. | 19. $A = s^2$. |
| 7. $2m + 3$. | 20. $N = s^2h$. |
| 8. $a \div 2$. | 21. $A = \frac{22}{7} r^2$. |
| 9. $4 \times t = 8$. | 22. $r = \frac{d}{t}$. |
| 10. $3f = 8$. | 23. $d = \frac{1}{2} gt^2$. |
| 11. $c - 5 = 8$. | 24. $v = lwh$. |
| 12. $5x + 2 = 12$. | 25. $V = \frac{22}{7} r^2h$. |
| 13. $3n - n = 18$. | |

Origin of symbols. Many of the symbols that are in common use in algebra at the present time have histories which not only are interesting in themselves but which also serve to indicate the slow and uncertain development of the subject. It is often found that symbols which seem without meaning represent some abbreviation or suggestion long since forgotten, and that operations and methods which we find hard to master have sometimes required hundreds of years to perfect.

In the early centuries there were practically no algebraic symbols in common use; one wrote out in full the words *plus*, *minus*, *equals*, and the like. But in the sixteenth century several Italian mathematicians used the initial letters *p* and *m* for + and - . Some think that our modern symbol - came into use through writing the initial *m* so rapidly that the curves of the letter gradually flattened out, leaving finally a straight line. The symbol + may have originated similarly in the rapid writing of the letter *p*. In the opinion of others these symbols were first used in the German warehouses of the

fifteenth century to mark the weights of boxes of goods. If a lot of boxes, each supposed to weigh 100 pounds, came to the warehouse, the weight would be checked, and if a certain box exceeded the standard weight by 5 pounds, it was marked $100 + 5$; if it lacked 5 pounds, it was marked $100 - 5$. Though the first book to use these symbols was published in 1489, it was not until about 1630 that they could be said to be in common use.

Both the symbols for multiplication given in the text were first used about 1630. The cross was used by two Englishmen, Oughtred and Harriot, and was probably an adaptation of the letter x , which is found some years earlier. The dot is first found in the writings of the Frenchman Descartes. It is interesting to note that Harriot was sent to America in 1585 by Sir Walter Raleigh, and returned to England with a report of observations. He made the first survey of Virginia and North Carolina, and constructed maps of those regions.

It is strange that the line was used to denote division long before any of the other symbols here mentioned were in use. This is, in fact, one of the oldest signs of operation that we have. The Arabs, as early as A.D. 1000, used both $\frac{a}{b}$ and a/b to denote the quotient of a and b . The symbol $\div$ did not occur until about 1630.

Equality has been denoted in a variety of ways. The word *equals* was usually written out in full until about the year 1600, though the two sides of an equation were written one over the other by the Hindus as early as the twelfth century. The modern sign $=$ was probably introduced by the Englishman Recorde, in 1557, because, he says, "Noe. 2. thynges can be moare equalle" than two parallel lines. This symbol was not generally accepted at first, and in its place the symbols $\parallel$, α , and ∞ are frequently met during the next fifty years.

EXERCISES

Use symbols to indicate the following numbers and operations with numbers:

- | | |
|------------------------------|------------------------|
| 1. Three increased by two. | 3. A less six. |
| 2. Twelve decreased by five. | 4. C increased by 8. |

5. Two times a .
6. One increased by 5 times x .
7. M added to 7.
8. 2 plus a .
9. Four minus 2 times y .
10. Area of a rectangle equals length times width.
11. Area of a triangle equals one half the product of the base and altitude.
12. Area of a circle equals $\frac{22}{7}$ times the square of its radius.
13. Distance equals speed times the time.
14. Perimeter of a rectangle equals twice the length plus twice the width. $l = \dots + 2w$
15. Average of two numbers equals their sum divided by 2.
16. Area of a square equals the square of its side.
17. Volume of a rectangular solid equals the product of the length, breadth, and thickness. $v = l \times b \times t$
18. Volume of a cylinder equals the area of its base multiplied by its altitude. $v = a \times h$
19. A certain number divided by 3 is 7 with a remainder of 2. $\frac{x}{3} = 7 + \frac{2}{3}$

4. **Algebraic expressions.** A letter or a group of number symbols involving one or more letters which stand for numbers is called an *algebraic expression*.

Thus, if n stands for some number, $7n$, $n + 2$, $5n - 1$, $\frac{n}{2}$, etc. are algebraic expressions.

Algebraic expressions in which the letters stand for the measures of distances, volumes, intervals of time, or other quantities are often called *formulas*.

Since each letter in an algebraic expression or formula stands for some number, it follows that the entire expression stands for some number and depends for its value on the values of the several letters involved.

Thus, if $a = 2$, then the algebraic expression $a + 5$ stands for 7; but if $a = 9$, then $a + 5$ stands for 14.

Again, if $m = 4$ and $n = 6$, then mn stands for 24, and $mn - 2$ stands for 22; but if $m = 5$ and $n = 7$, then mn stands for 35 and $mn - 2$ stands for 33.

ORAL EXERCISES

1. What number does $3d$ represent if $d = 10$? if $d = 100$?
2. What number does $5x + 2$ represent if $x = 1$? if $x = 3$?
3. What number does $2d - 7$ represent if $d = 4$? if $d = 10$?
4. What number does $l + w$ represent if $l = 5$ and $w = 2$? if $l = 10$ and $w = 8$?
5. What number does $a + b + c$ represent if $a = 12$, $b = 5$, and $c = 9$? if $a = 8$, $b = 15$, and $c = 10$?
6. The fact that p , the distance around a square, is four times the length of one of the sides, s , is stated by the expression $p = 4s$. Find p , if $s = 3$ inches; if $s = 5$ inches; if $s = 12$ inches; if $s = 15$ feet; if $s = 21\frac{1}{2}$ feet.
7. The fact that the perimeter of a rectangle is the sum of twice the length increased by twice the width is expressed by $p = 2l + 2w$. Find the perimeter of a rectangle which is 4 inches long and 3 inches wide. Find p if $l = 4$ inches and $w = 6$ inches; if $l = 9$ inches and $w = 8$ inches; if $l = 10$ feet and $w = 1$ foot; if $l = 18$ feet and $w = 1\frac{1}{2}$ feet.
8. A classroom is 28 feet by 36 feet. Find its perimeter. If it were 1 foot longer and 5 feet wider what would be its perimeter?

9. If p is the perimeter of a triangle and the sides are a , b , and c respectively, then $p = a + b + c$. Find the perimeter of a triangle whose sides are 3 inches, 4 inches, and 6 inches respectively. Find p if $a = 10$ inches, $b = 5$ inches, and $c = 12$ inches; if $a = 2$ feet, $b = 5$ feet, and $c = 3\frac{1}{2}$ feet.

10. The fact that the area of a rectangle is the product of the length and width is expressed by $A = lw$, where these dimensions are expressed in the same unit of measure. Find the area of a rectangle which has a length of 8 inches and a width of 5 inches. Find A if $l = 10$ inches and $w = 4$ inches; if $l = 4$ feet and $w = 2$ feet 6 inches; if $l = 20$ rods and $w = 40$ rods; if $l = 6$ feet and $w = 27$ inches.

11. Since the interest earned equals the product of the principal invested and the rate paid, then $i = pr$. Find i if $p = \$400$ and $r = 0.06$.

12. Compute the corresponding values of i for the given values of p and r in $i = pr$ and complete the following table:

$$i = pr$$

If $p =$	\$200	\$100	\$150	\$275	\$620	\$1220	\$2000
and $r =$	5%	6%	5%	7%	$6\frac{1}{2}\%$	$4\frac{1}{4}\%$	$4\frac{3}{4}\%$
then $i =$	\$10	?	?	9.2	40.30	50.5	85.00

13. The fact that the area of a square equals the square of its side is expressed by $A = s \times s$ or $A = s^2$. Complete the following table, giving the correct values for A for each of the several values of s :

$$A = s^2$$

If $s =$	3 in.	5 in.	$3\frac{1}{2}$ in.	12 in.	2 ft.	5 yd.	$7\frac{2}{3}$ rd.	50 mi.
then $A =$	9 sq. in.	25	12.25	144	4	25	58.33	2500

14. The fact that the volume of a rectangular solid, such as a chalk box, a block of ice, or a rectangular room, is the product of the length, width, and height of the solid is stated as $V = lwh$. Compute the volumes of the rectangular solids for the given values of the dimensions and complete the following table:

$$V = lwh$$

If $l =$	3 in.	5 in.	7 in.	6 in.	5 ft.	8 ft.	3 ft.
and $w =$	8 in.	6 in.	10 in.	12 in.	8 ft.	$2\frac{1}{2}$ ft.	10 in.
and $h =$	2 in.	4 in.	12 in.	$1\frac{1}{2}$ in.	2 ft.	4 ft.	15 in.
then $V =$	48 cu. in.	120	840	?	80	80	540

15. The approximate distance which a falling body travels through in a given number of seconds is expressed by $d = \frac{1}{2} at^2$, in which d = the distance fallen in feet, a has the constant value 32, and t = the time in seconds. Find the distances fallen for each of the time intervals given below:

$$d = \frac{1}{2} at^2$$

$a,$	=	32	32	32	32	32	32	32
$t,$ in seconds	=	2	3	5	1	6	10	20
$d,$ in feet	=	64	?	?	52	192	320	?

5. Number representation through algebraic symbols. It has been shown that the value of an algebraic expression which involves one or more letters depends upon the values assigned to these letters. The fact that a given expression, such as $l \cdot w$, can represent the area of any rectangle or that $p \cdot r$ can represent any interest payment illustrates

one of the principal advantages of algebraic over arithmetic methods. The manner of expressing one number in terms of one or more other number symbols is illustrated below.

EXAMPLE

Write the expression for a number which is 2 greater than a given number, n .

Solution. Let n = the given number.

Then $n + 2$ = the required number.

EXERCISES

Write an algebraic expression for a number which is

1. 3 greater than b .
2. 5 less than r .
3. r less than 5.
4. Twice as great as b .
5. Five times as great as r .
6. One half as great as x .
7. Two thirds as great as x .
8. Three times d .
9. Equal to the product of h and b .
10. One half of the product of a and b .
11. Equal to the quotient of b divided by 5.
12. Five less than $3h$.
13. Two greater than the product of m and n .
14. Twelve less than the quotient of x divided by f .
15. Equal to the sum of ten times a and three times b .
16. Equal to the area of a rectangle whose base is b and whose altitude is a .
17. Equal to the area of a square whose side is s ; whose side is x .

18. Equal to the cubic units in the volume of a box whose length is l , whose width is w , and whose height is h ; whose length is 10, whose width is w , and whose height is h .

19. The speed of one automobile is two thirds that of another. Represent the speed of the slower if the faster moves 24 miles per hour; r miles per hour.

20. A boy is five years older than his brother. If b represents the age of the boy, represent the age of the brother.

21. A child deposits p cents in a savings bank each week. Another child deposits each week five cents less than the first child. How many cents does the second child deposit weekly?

22. If x represents a given number, what will represent a number one greater than x ? two greater than x ? three greater than x ?

Even integers are those exactly divisible by 2. Odd integers are those not exactly divisible by 2.

Consecutive integers are integers arranged in the natural order, like 4, 5, 6, 7, 8, etc. /

Consecutive odd integers are odd integers arranged in the natural order, like 5, 7, 9, 11, 13, etc. 2

Consecutive even integers are even integers arranged in the natural order, like 4, 6, 8, 10, 12, etc. 2

23. What is the difference between any two consecutive integers? between any two consecutive odd integers? between any two consecutive even integers?

24. If n is an integer, what is the next consecutive integer? If x is an integer, what are the next two consecutive integers?

25. If m is an odd integer, what is the next consecutive odd integer? the next two consecutive even integers?

26. If x is an odd integer, is $x + 1$ odd or even? If x is even, what of $x + 1$?

27. If x is an odd integer, what of $x + 3$? $x + 6$? If y is even, what of $y + 2$? $y + 3$?

28. If a is an integer, is $2a$ odd or even? Is $2a + 1$ odd or even?

29. Is the sum of two consecutive integers always odd?

HINT. Express the integers algebraically, add them, and divide by 2.

30. Is the sum of three consecutive integers always even?

31. By what number is the sum of three consecutive integers always divisible?

32. Write four consecutive integers the first of which is n ; $n + 2$; $n + 7$; four the last of which is $n + 12$; $n - 6$; $n - 3$.

33. Write a series of three consecutive even numbers. If n stands for an even number, what will represent the next two consecutive even numbers?

34. If n represents an odd number, what will represent the next two consecutive odd numbers?

35. If a boy is 17 years old at present, how old was he 2 years ago? n years ago? How old will he be 5 years hence? n years hence?

36. If a boy is y years old and his sister is twice as old as he, how old will she be 3 years hence? 7 years hence? How old was she three years ago? n years ago?

37. If n represents the number of nickels in a collection, what will represent their value in cents? their value in dimes? in dollars?

38. If q represents a number of quarters in a certain collection, what will represent their value in cents? in nickels? in half dollars?

39. If a collection is composed of d dimes and h half dollars, represent the value of the collection in cents; in nickels; in dimes; in dollars.

40. How much change should be received from a five-dollar bill in payment for an article costing \$2? n dollars?

6. Simplifying algebraic expressions. In the same way that it is possible to add two number symbols such as 5 and 3, just so it is possible to add two or more number symbols which involve the same literal parts.

Thus	$5t + 2t = 7t$
and	$3a + 4a - 1a = 6a$
and	$2x + 5 + x + 2 = 3x + 7.$

EXERCISES

Simplify the following by collecting the like number symbols:

1. $5 + 2.$ 5. $5r - 2r + 3.$ 9. $5x - 6x + 3x.$
2. $8 - 5.$ 6. $2h + 10h - 2.$ 10. $2 - m + 7 + 5m.$
3. $4s + 2s.$ 7. $12h + 10h - 2.$ 11. $5a + b - 3a.$
4. $5h - 3h.$ 8. $3 + 2a + 5a - 2.$ 12. $3x - 2y - 7y - x.$
13. $4m + 7n - 2n.$ 17. $4h + 7h = 33 - 11.$
14. $3a + 2a = 10.$ 18. $6r - 5r = 1 + 8.$
15. $5r - 3r = 8.$ 19. $k + 8k = 20 - 2.$
16. $12x - 5x = 18 + 3.$ 20. $s - 8s + 12s = 32 - 7.$

7. Solving equations. A statement of equality between two equal numbers or number symbols is called an *equation*.

Thus $3f = 36$ is an equation, for it is obviously true when $f = 12$.

Also $5t + 2 = 12$ is an equation, for it is true when $t = 2$.

The process of determining the particular value of the letter for which the statement is true is called *solving the equation*. Skill and accuracy in solving equations are fundamental to the study of algebra.

EXAMPLES

1. Solve for d ,

$$10d = 250.$$

Solution.

$$10d = 250.$$

Dividing by 10,

$$d = 25.$$

Check. Substituting 25 for d in $10d = 250$,

$$10 \times 25 = 250,$$

or

$$250 = 250.$$

2. Solve for m ,

$$2m + 3m = 35 + 5.$$

Solution.

$$2m + 3m = 35 + 5. \quad (1)$$

Collecting,

$$5m = 40.$$

Dividing by 5,

$$m = 8.$$

Check. Substituting 8 for m in (1),

$$2 \times 8 + 3 \times 8 = 35 + 5,$$

or

$$40 = 40.$$

EXERCISES

Solve each of the following equations for the number represented by the letter involved:

1. $2n = 10.$

5. $3x = 35 - 2.$

9. $8n = 60 - 4.$

2. $3n = 21.$

6. $7r = 47 + 2.$

10. $15p = 50 - 5.$

3. $5n = 20.$

7. $9s = 15 + 30.$

11. $2n + 3n = 15.$

4. $4a = 20 + 8.$

8. $12m = 38 - 2.$

12. $5r - 2r = 30.$

13. $3h + 7h = 100.$

17. $5x - 3x = 12 - 4.$

14. $12h - 5h = 56.$

18. $x + x + 3x = 2 + 8.$

15. $3m - m = 12.$

19. $h + 3h + 2h = 48.$

16. $2b + 3b = 6 + 9.$

20. $5m - m + 3m = 70 - 7.$

8. Use of symbols in problem solving. By means of algebraic symbols and equations involving number symbols a simple and direct method is provided for the solution of certain types of problems.

In order that problems may be expressed in form suitable for algebraic treatment, it is often necessary to translate a verbal statement containing number relationships into an expression involving algebraic symbols. The ability to do this is one of the most important results of the study of algebra.

EXAMPLE

Find two numbers such that one is four times the other and their sum is 110.

Solution. Greater number + less number = 110.

Let l = the less number.

Then $4l$ = the greater number.

$$4l + l = 110.$$

$$5l = 110.$$

Dividing by 5, $l = \frac{110}{5} = 22$, the less number.

Then $4l = 4 \times 22 = 88$, the greater number.

Check. Testing in the conditions set by the problem,

$$88 = 4 \times 22$$

and

$$88 + 22 = 110.$$

PROBLEMS

1. The sum of two numbers is 72. One number is 5 times the other. Find the numbers.

2. Two boys together have 40 dollars. The first has 7 times as much as the second. How many has each?

3. The first of three numbers is twice the second, and the second is twice the third. Their sum is 77. What are the numbers?

HINT. Let x equal the third number.

4. Three boys together own 50 chickens. The first has four times as many as the third, and the second has five times as many as the third. How many has each?

5. Three newsboys together sold 108 papers. The first sold three times as many as the second, who sold twice as many as the third. How many did each boy sell?

6. What is the area of a triangle whose base is twice as long as the altitude, and the sum of whose base and altitude is 18 inches?

HINT. $A = \frac{1}{2} \text{ base} \times \text{altitude}.$

7. The perimeter of a rectangle is 200 feet. The length is four times the width. Find the dimensions.

8. The perimeter of a rectangle is 96 feet. It is 7 times as long as it is wide. Find the dimensions.

9. The perimeter of a certain square is 116 inches. What is the length of one side?

10. What is the side of a square whose perimeter is 256 feet? of one whose perimeter is 320 feet?

11. The perimeter of a rectangle formed by placing two equal squares side by side is 72 inches. Find the side of each square.

72
1-2-4-6

12. The perimeter of a rectangle formed as in Problem 11 is 432 inches. What is the side of the square?

13. A rectangle is three times as long as it is wide. Its area is 147 square feet. Find its dimensions.

14. There are four numbers whose sum is 288. The second number is twice the first, the third number is 3 times the first, and the fourth is 3 times the second. What are the numbers?

15. The perimeter of a certain rectangle is 460 feet. It is 4 times as long as it is wide. Find its dimensions.

16. Three equal squares are placed side by side; the perimeter of the rectangle thus formed is 64 feet. What are the dimensions of the squares?

17. Four equal squares are placed together so as to form another square whose perimeter is 96 feet. What is the side of each square?

9. Terms. Algebraic expressions are often regarded as made up of parts separated by the signs of operation $+$ or $-$. Each of these parts is called a *term*.

Thus, $3n$ is an expression of one term; $n + 3$, of two terms; $na + 2b - c$, of three terms, etc. Each of these expressions is an algebraic symbol for a number.

10. Factors. A *factor* of a product is any one of the number symbols which when multiplied together form the product.

Thus $2mn$ means 2 times m times n . Here 2, m , and n are each factors of $2mn$.

Again, $4(l + w)$ means 4 times the sum of l and w . Here the number 4 and $l + w$ are factors of the expression.

EXERCISES

1. Name the factors in 3×5 ; in $2 \times 7 \times 11$; in $5ab$; in lwh ; in $\frac{1}{2}ba$; in $\frac{1}{3}Ba$; in $a(x+r)$; in $h(b+a)$.

Find the value of each of the following expressions for the following values of the factors: $a = 6$, $b = 2$, $n = 7$, $l = 4$, $w = 2$, and $x = 9$.

2. $5a^{30}$ 3. $3n^{21}$ 4. ab^{17} 5. lw^{40} 6. $\frac{1}{2}ba$ 7. nab

8. $\frac{a}{w}$

12. $ba \div 2$

17. $3x + w - l + 1$ 26

9. $\frac{l}{w} + 1$

13. $a + b + w$

18. $4w - l + 3x - b$ 27

10. $5nlw + 3$

14. $3a - n + x$

19. $\frac{a}{b} + \frac{l}{w}$ 5

15. $a - b + n - 1$

11. $2l + 2w$

16. $ab - n + wx$ 3

20. $\frac{a}{2} - \frac{x}{3}$ 0

11. **Exponents.** An *exponent* is a positive integer written at the right and a little above another number to indicate how many times the number is to be used as a factor.

(Later this definition will be extended to include fractions and other numbers as exponents.)

Thus, 3^2 means 3×3 ; 2^4 means $2 \times 2 \times 2 \times 2$; b^3 means $b \times b \times b$; and $3s^2$ means $3 \times s \times s$.

If a number is used once as a factor its exponent is 1, as in $2a^1$. This exponent is usually not written.

ORAL EXERCISES

1. What operations are indicated in 5^2 ? in 3^4 ? in 3×7^2 ? in 3×5^3 ? in $3^2 \times 2^4$?

2. Name the exponents used in the several terms of Exercise 1.

3. Name the exponents in $2a^2$, $5x^3$, $3c$, $2m^2n^3$; in $3x^2 \div 1$, $ab \times 2$, s^2h .

In the following substitute 2 for m and 5 for n and find the value of each expression :

4. m^2 .

9. $m^3 + 5$.

14. $5 + m^2 + n^2$.

5. $m^2 + 1$.

10. $25 + n^2$.

15. $m^2 + mn + n^2$.

6. $m^2 + n$.

11. m^2n^2 .

16. $3 - m + n^2$.

7. $m^2 + n^2$.

12. $m^2n + 1$.

17. $n^2 - m^2 - nm$.

8. $2n^3$.

13. $3mn^2 - 2$.

18. $m^3 + n^3 - 3$.

12. Coefficients. If a number is the product of two or more factors, either of these factors is called the *coefficient* of the product of the others.

Thus, in 5×3 , 5 is the coefficient of 3, and 3 is the coefficient of 5. In $3a^2x$, 3 is the coefficient of a^2x , x is the coefficient of $3a^2$, and a^2 is the coefficient of $3x$.

The numerical coefficient 1, as in $1a$, $1mn$, etc., is usually omitted, but is understood.

ORAL EXERCISES

Find the value of the following :

1. $4h^2$ if $h = 5$; if $h = 3$.

2. hk^2 if $h = 2$ and $k = 3$.

3. $5m^3$ if $m = 2$; if $m = 1$; if $m = 5$.

4. rs^3 if $r = 2$ and $s = 5$.

5. $2lw^2$ if $l = 5$ and $w = 4$.

6. $3a^2b$ if $a = 2$ and $b = 3$; if $a = 5$ and $b = 2$.

7. Read the numerical coefficients in the terms of the preceding.

8. Read the exponents of each factor in the terms of the preceding.

13. Parentheses and radical signs. If two or more terms connected by signs of operation are inclosed in parenthesis, the entire expression is treated as a single number symbol.

Thus $2(5 + 3)$ means 2×8 , or 16; $(6 + 3) \times (5 - 2)$ means 9×3 , or 27; $(7 - 1) \div 2$ means $6 \div 2$, or 3; $(2 + 5)^2$ means 7^2 , or 49; $3(a^2 + b^2)$ means 3 times the sum of a^2 and b^2 .

As in arithmetic, the symbol for square root is $\sqrt{\quad}$, and that for cube root is $\sqrt[3]{\quad}$.

The name *radical sign* is applied to all symbols like the following: $\sqrt{\quad}$, $\sqrt[3]{\quad}$, $\sqrt[4]{\quad}$. The small figure in the radical sign, like 3 in $\sqrt[3]{\quad}$, is called the *index* of the radical.

ORAL EXERCISES

Find the value of

1. $3(2 + 5)$.
2. $5(6 - 1)$.
3. $(2 + 1)(3 + 4)$.
4. $(3 + 2)(9 - 7)$.
5. $(9 - 3) \div 3$.
6. $(12 - 8) \div (6 - 4)$.
7. $(a + x)^2$ if $a = 6$ and $x = 1$.
8. $(m - n)^3$ if $m = 9$ and $n = 4$.
9. $\sqrt{9} + \sqrt[3]{8}$.
10. $\sqrt{6 + 3}$.
11. $2^2 + \sqrt{4}$.
12. $6 + 3^2$.
13. $4^2 - \sqrt{4}$.
14. $\sqrt{3^2 + 4^2}$.
15. $\sqrt[3]{5^2 + 2}$.
16. $\sqrt{(2 + 5)(8 - 1)}$.
17. $\sqrt[3]{3(2 + 1)(7 - 4)}$.
18. $\sqrt{n} + x - y$ if $n = 25$, $x = 4$, and $y = 6$.
19. $(\sqrt{x} - 3) + y$ if $x = 49$ and $y = 2$.

NOTE. There has been a considerable variety in the symbols for the roots of numbers. The symbol $\sqrt{\quad}$ was introduced in 1544 by the German, Stifel, and is a corruption of the initial letter of the Latin word *radix*, which means "root." Before his time square root was denoted by the symbol $\mathbb{R}$, used nowadays by physicians on prescriptions as an abbreviation for the word recipe. Thus $\sqrt[4]{5}$ would have been denoted by $\mathbb{R}^4 5$. Some early writers used a dot to indicate square root, and expressed $\sqrt{2}$ by $\cdot 2$.

14. Order of fundamental operations. In evaluating such an expression as $36 \div 3 - 2 \times 6$, the numbers 12, 10, and a final result of 60 are obtained if the operations are performed in the order in which they occur. If, however, the indicated division and multiplication are performed before the indicated subtraction, the result is 12 minus 12, and a final result of 0 is obtained. These results show that the value of numeric expressions may depend on the order in which the indicated operations are performed. It is customary to observe the following

RULE. *In a series of operations involving addition, subtraction, multiplication, and division of arithmetical numbers, the multiplications and divisions shall be performed first in the order in which they occur.*

The additions and subtractions in the resulting expression shall then be performed in the order in which they occur or in any other order.

If parentheses occur, each expression within a parenthesis should first be simplified in accordance with the preceding rule and the rule then applied to the entire expression.

EXAMPLES

Simplify :

1. $18 \div 2 + 5 - 4 \times 2$.

Solution. $18 \div 2 + 5 - 4 \times 2 =$
 $9 + 5 - 8 = 6.$

2. $24 \div 8 \cdot 2 - 4(7 - 3) \div 8 + 2(5 + 6 \cdot 3).$

Solution. $24 \div 8 \cdot 2 - 4(7 - 3) \div 8 + 2(5 + 6 \cdot 3) =$
 $6 - \qquad \qquad 2 + \qquad 46 \qquad = 50.$

EXERCISES

Simplify the following:

1. $20 - 5 + 6 - 8$ 13
2. $16 + (8 + 3)$ 27
3. $12 - (5 + 2)$ 5
4. $9 - (7 - 3)$ 5
5. $14 - (16 - 7) + 12 - 5$ 12
6. $6 \div 2 + 2$ 5
7. $8 \cdot 6 \div 4 - 10$ 2
8. $24(2 + 3)$ 120
9. $23 - 2 \cdot 6 - 8 \div (5 - 1)$ 7
10. $(10 - 3) \cdot (16 - 3 \cdot 3 + 8 \div 2)$ 77
11. $(14 - 3)(16 \div 4 \cdot 5) + 8 \cdot 2$ 36
12. $(16 - 6)(17 - 7) \div 100 \cdot 5 + 13$ 18
13. $3^2 + 2 \cdot 3 + 2^2$ 19
14. $8^2 - 2 \cdot 7 \cdot 3 + 3^3$ 49
15. $3 \cdot 4 - 6 \cdot 0 + 3 \times 5 \div 2^2 \times 7 - 3^2$ 29 1/2
16. $2^2 \cdot 3^2 - 2 \times 2 \times 3 \times 5 + 5^2$ 7

Find the value of

17. $(a + b)^2$ if $a = 2$ and $b = 3$; if $a = 5$ and $b = 2$. 25, 4
18. $(x - y)^2$ if $x = 5$ and $y = 2$; if $x = 6$ and $y = 4$. 9, 4
19. $A^2 - 2A - 8$ if $A = 10$; if $A = 8$; if $A = 12$. 72, 112
20. $m^2 + m - 110$ if $m = 12$; if $m = 10$. 110, 10
21. $x^2 + (a - b)^2$ if $x = 8$, $a = 10$, and $b = 6$. 80
22. $a^2 + \sqrt{b^2 + c^2}$ if $a = 6$, $b = 3$, and $c = 4$. 41
23. $(h - k)^2 - \sqrt{h^2 - k^2}$ if $h = 5$ and $k = 3$. 0

CHAPTER II

GRAPHICAL REPRESENTATION

15. Use of graphs. One often finds it difficult to grasp the important facts indicated by a column of figures, even though the meaning of each individual number is perfectly understood. When, however, these numbers are presented in the form of a diagram or picture, the facts frequently stand out in a clear and striking manner. Such a pictorial representation made for the purpose of comparing several similar or related quantities is called a *graph*.

Today one finds graphs in newspapers, in popular magazines, in scientific journals, and in social, industrial, and business reports. The ability to grasp quickly the important points presented by a graph — that is, the ability to comprehend a graph — is necessary for all who would read with understanding. Furthermore, the ability to present statistical data, whether of a scientific or a commercial nature, effectively by means of a graph is often very desirable.

16. Types of graphs. There are many ways of comparing quantities by means of graphs, but only three types will be considered here. These three types illustrate the most important methods of graphic presentation and comparison. They are:

1. The *bar graph*, in which the comparison is shown by parallel bars of the same width, but with lengths proportional to the quantities represented.

2. The *broken-line graph*, drawn through points located at distances from the base line proportional to the quantities they represent.

3. The *circle graph*, in which the relative sizes of the quantities compared are represented by sectors of the same circle.

These methods will be treated in the order mentioned above.

17. The bar graph. In this device the several numbers or quantities to be compared are represented by bars or heavy lines. This type of graph is well adapted to the comparison of quantities or magnitudes which are similar but which are not necessarily related to one another.

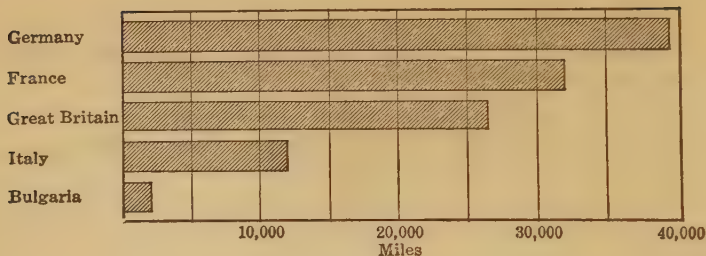
EXAMPLE

The railroad mileage in each of several European countries, to the nearest 500 miles, is as follows:

Germany	39,500 miles
France	32,000 miles
Great Britain.	26,500 miles
Italy	12,000 miles
Bulgaria	1,500 miles

Indicate the relationship between these values by means of the bar graph.

Solution. It will be convenient to let one of the horizontal divisions represent the length of five thousand miles of railway. Then we can draw a bar for Germany 7.9 units in length, for France 6.4 units, for Great Britain 5.3 units, for Italy 2.4 units, and for Bulgaria .3 unit in length. If we make each of the bars one half unit in width we shall get the graph shown on the following page.



EXERCISES

Use bar graphs to show the following relationships:

1. The basins of the following rivers contain the indicated number of square miles to the nearest 100,000:

Mississippi and tributaries	1,300,000
Yukon	300,000
Columbia	300,000
Amazon	2,500,000
Nile	1,100,000

2. The naval expenditures to the nearest \$1,000,000 of the Great Powers for the year 1921-1922 were as follows:

Great Britain	\$406,000,000
United States	426,000,000
France	182,000,000
Italy	81,000,000
Japan	249,000,000

3. In 1922 various crops were produced in the United States in the following amounts to the nearest 100 million bushels:

Corn	3700 million bushels
Wheat	3000 million bushels
Oats	3000 million bushels
Barley	1000 million bushels
Rye	800 million bushels

4. The average monthly production of cement to the nearest 100 thousand barrels in the United States for a period of years was as follows:

YEAR	THOUSANDS OF BARRELS	YEAR	THOUSANDS OF BARRELS
1913	7,700	1919	6,700
1914	7,400	1920	8,300
1915	7,100	1921	8,200
1916	7,600	1922	9,500
1917	7,700	1923	11,500
1918	5,900		

QUERY. How did the slump in the building trades, which lasted from 1918 to 1920, affect the cement production in the United States?

18. The broken-line graph. This type of graph is particularly effective for representing the relation between two quantities or magnitudes. This type of graph is frequently used for the representation of changes in some quantity taking place during some particular interval of time.

EXAMPLE

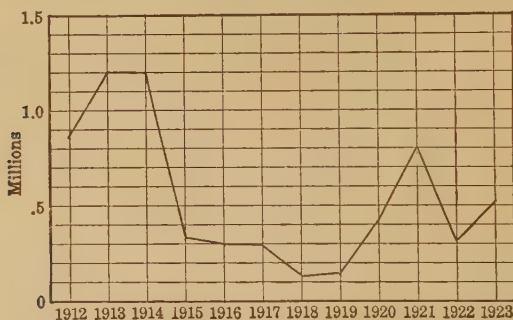
Plot the data contained in the following table:

NUMBER OF IMMIGRANTS ARRIVING IN UNITED STATES, 1912-1923

1912	840,000	1918	110,000
1913	1,200,000	1919	140,000
1914	1,220,000	1920	430,000
1915	330,000	1921	810,000
1916	300,000	1922	310,000
1917	300,000	1923	520,000

Solution. Let one space along the base line represent an interval of one year and let one space along the vertical line at the left represent 100,000 immigrants.

Then the following graph shows the trend of immigration over the period :



QUERIES. What effect did the outbreak of the European war apparently have on the immigration into the United States?

Did the entry of the United States into the war have any noticeable effect on the curve?

EXERCISES

Use broken-line graphs to show the following data :

1. The number of motor cars to the nearest 10,000 (both passenger cars and trucks) registered in the United States on January 1 of each of several years was as follows :

YEAR	NUMBER OF CARS	YEAR	NUMBER OF CARS
1915	2,450,000	1920	9,230,000
1916	3,510,000	1921	10,470,000
1917	4,980,000	1922	12,240,000
1918	6,150,000	1923	13,000,000
1919	7,570,000	1924	15,280,000

QUERIES. Assuming that the trend during the next year is the same as that over the past several years, what will be the approximate number of cars in the United States in 1925?

Did the World War have any effect on the registration of cars?

2. The population of Philadelphia to the nearest 1000 for several census dates was as follows :

YEAR	POPULATION	YEAR	POPULATION
1860	566,000	1900	1,294,000
1870	674,000	1910	1,549,000
1880	847,000	1920	1,824,000
1890	1,047,000		

3. The cotton production in the United States to the nearest 100,000 bales for a ten-year period was as follows :

YEAR	THOUSANDS OF BALES	YEAR	THOUSANDS OF BALES
1912	13,700	1917	11,300
1913	14,100	1918	12,000
1914	16,100	1919	11,400
1915	11,200	1920	13,400
1916	11,500	1921	8,300

4. The number, to the nearest 100, of secondary schools in the United States reporting to the commissioner of education was as follows :

YEAR	1890-1891	1900-1901	1910-1911	1919-1920
Secondary schools in United States	4500	8200	12,200	16,400

5. Enrollments in the secondary schools of the United States were as follows :

YEAR	1890-1891	1900-1901	1910-1911	1919-1920
Number of pupils	310,000	650,000	1,115,000	2,041,000

QUERY. From the trends shown in the graph, estimate the approximate number of pupils there will be in the public secondary schools in 1930.

6. The depth of a river was measured every five feet from bank to bank, giving the following records: 0, 1 foot, 2 feet 6 inches, 4 feet, 7 feet 9 inches, 10 feet, 15 feet, 16 feet 6 inches, 16 feet, 17 feet, 18 feet 8 inches, 17 feet 3 inches, 15 feet, 14 feet 6 inches, 12 feet, 10 feet 9 inches, 10 feet, 8 feet 3 inches, 6 feet, 5 feet 9 inches, 3 feet, 2 feet 9 inches, 1 foot 3 inches, 0. Plot, using a line graph.

QUERY. If these quantities were plotted as distances below the horizontal axis, what would the appearance of the curve suggest?

7. The enrollments, to the nearest 1000, reported for California secondary schools for a period of years were as follows:

SCHOOL YEAR	NUMBER OF PUPILS	SCHOOL YEAR	NUMBER OF PUPILS
1912-1913	58,000	1917-1918	125,000
1913-1914	66,000	1918-1919	137,000
1914-1915	76,000	1919-1920	162,000
1915-1916	95,000	1920-1921	191,000
1916-1917	113,000	1921-1922	225,000

8. The gold coin held in the United States during a given period was as follows:

YEAR	VALUE IN MILLIONS OF DOLLARS	YEAR	VALUE IN MILLIONS OF DOLLARS
1913	1870	1919	3110
1914	1890	1920	2710
1915	1990	1921	3300
1916	2450	1922	3790
1917	3020	1923	4210
1918	3080		

QUERY. What effects of the following occurrences are indicated on this curve? (1) the outbreak of the World War; (2) the entrance of the United States into the war; (3) the close of hostilities in 1918; (4) the industrial depression in 1920.

9. The average price of gas in dollars per thousand cubic feet for fifty-one cities in the United States is given in the following table :

YEAR	PRICE	YEAR	PRICE	YEAR	PRICE
1913	\$0.95	1917	\$0.92	1921	\$1.32
1914	0.94	1918	0.95	1922	1.28
1915	0.93	1919	1.04	1923	1.26
1916	0.92	1920	1.09		

10. The values, in millions of dollars, of the exports from the United States and from England for a given period were as follows :

YEAR	UNITED STATES	ENGLAND	YEAR	UNITED STATES	ENGLAND
1913	2400	2600	1919	7800	3600
1914	2100	2100	1920	8100	4800
1915	3500	1800	1921	4400	2700
1916	5400	2400	1922	3800	3200
1917	6200	2500	1923	4200	3500
1918	6000	2400			

Plot on the same diagram a solid line for the exports of the United States and a broken line for the exports of England.

QUERIES. Point out the effect of the war on the trade of the United States in relation to the trade of England.

Does the relation seem to be returning to that which prevailed before the war ?

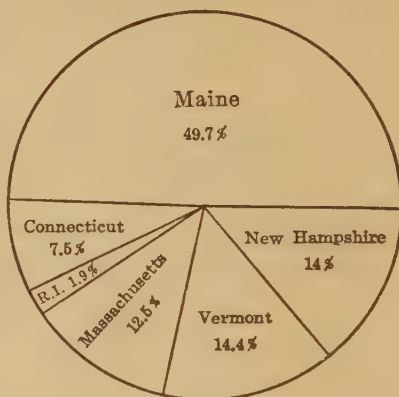
19. Circle graphs. When it is desired to show the relation of several quantities to one another and to the whole of which they are parts, the circle graph is useful.

The method of making a circle graph is illustrated on the following page :

EXAMPLE

The areas of the states of Maine, New Hampshire, Vermont, Massachusetts, Rhode Island, and Connecticut are 33,040, 9341, 9564, 8266, 1250, and 4965 square miles respectively. Represent the relation between these areas by a circle graph.

Solution. The sum of the areas of the six states is 66,426 square miles. The percentages of this total area for the various states are Maine 49.7%, Rhode Island 1.9%, Connecticut 7.5%, Massachusetts 12.5%, New Hampshire 14.0%, Vermont 14.4%.



Since there are 360° around the center of a circle, the number of degrees representing the area of Maine is 49.7% of 360° , or 178.9° ; Rhode Island, 1.9% of 360° , or 7° ; Connecticut, 7.5% of 360° , or 27° ; Massachusetts, 12.5% of 360° , or 45° ; New Hampshire, 14.0% of 360° , or 50° ; Vermont, 14.4% of 360° , or 52° .

These figures may be given in tabular form as follows:

STATE	AREA	PER CENT	ANGLE
Maine	33,040 square miles	49.7	179°
New Hampshire .	9,341 square miles	14.0	50°
Vermont	9,564 square miles	14.4	52°
Massachusetts . .	8,266 square miles	12.5	45°
Rhode Island . . .	1,250 square miles	1.9	7°
Connecticut . . .	4,965 square miles	7.5	27°

EXERCISES

1. The area of Europe is about 3,900,000 square miles and is divided as follows:

Cultivated lands	27 %
Prairies	24 %
Forests	28 %
Unproductive lands	21 %

Plot, using circle graph.

2. The area of North and South America is 17,550,000 square miles. Of this area 18% is mountains, frigid and other unproductive lands; 30%, forest; 40%, prairies, pampas, and savannahs; 12%, cultivated lands. Plot these statistics, using circle graph.

3. The items of expense of a certain family have the relative importance shown below. Indicate this situation by means of the circle graph.

Food	43.1 %
Shelter	17.7 %
Clothing	13.2 %
Fuel, heat, etc.	5.6 %
Sundries	20.4 %
Total	<u>100.0 %</u>

4. The population of the earth, to the nearest 1,000,000 people, is divided, according to races, approximately as follows:

Caucasian (white)	821,000,000
Mongol (yellow)	645,000,000
Semitic	75,000,000
Negro	139,000,000
Other races	68,000,000

Indicate this distribution by means of the circle graph.

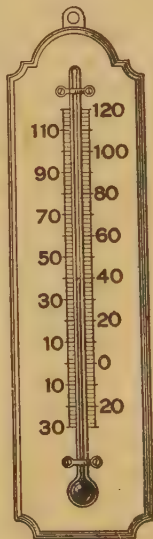
CHAPTER III

POSITIVE AND NEGATIVE NUMBERS

20. **Meaning of positive and negative numbers.** On the thermometer diagram shown at the right the reading is 40° above zero. If the temperature should drop 25° , the thermometer reading would be $40^{\circ} - 25^{\circ}$, or 15° . If it should drop 70° , then the reading would be $40^{\circ} - 70^{\circ}$, or 30° below zero, which is indicated by -30° , etc. A record such as -30° means that the reading shows a temperature which is 30° below that reading which is called zero degrees, or 0° . In contrast to the readings below zero, such as -10° , -15° , -20° , etc., the readings above zero are indicated by prefixing plus signs to the numerical records, as in $+10^{\circ}$, $+25^{\circ}$, $+43^{\circ}$, etc.

A similar method is used in recording latitude readings, those in the Northern Hemisphere being plus, or positive, readings, and those in the Southern Hemisphere being minus, or negative, readings. Thus latitude 10° north is expressed $+10^{\circ}$ and latitude 10° south is expressed -10° .

Again amounts of money on deposit in the bank may be regarded as positive and the amounts of overdrafts may be regarded as negative.



21. Illustrations of positive and negative numbers. In general, the uses made of positive and negative numbers arise in connection with measures of quantity which may be regarded as existing in opposite senses; as, for example, money on deposit in a bank and overdrawn accounts, distances measured in opposite directions from a fixed point, time measured before and after a fixed date, etc.

Any rise in the thermometer reading, or upward change, is indicated by placing the sign $+$ before the number of degrees indicating the amount of change; thus, a change of $+15^{\circ}$ in temperature means a rise of 15° .

Similarly, a fall in the thermometer reading, or downward change, is indicated by placing the sign $-$ before the number of degrees indicating the change; thus, a change of -10° in temperature means a fall of 10° .

From the above it is evident that to indicate a $+15^{\circ}$ change, that is, to add 15° to a given reading, we count up 15° , and to indicate a -15° change we count down 15° , from the given reading.

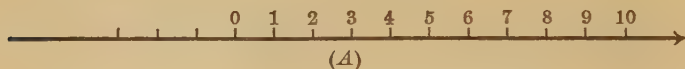
Similarly, the latitude reached by sailing north 20° is indicated by adding $+20^{\circ}$ to the given latitude reading, and sailing south 20° is indicated by counting down or south 20° , that is, by adding -20° to the given latitude reading.

Likewise, the increasing of one's deposits at the bank is equivalent to adding positive sums to the account, and taking out money from the bank is equivalent to adding withdrawals or negative sums of money.

$$\text{Thus,} \qquad \$25 + (\$ + 15) = \$ + 40,$$

$$\text{and} \qquad \$100 + (\$ - 25) = \$ + 75.$$

22. Addition and subtraction by use of a scale. Let us suppose that equal distances are taken on a line and that the successive points of division are marked with the natural, or positive, numbers as follows :



Such a scale of numbers may be used to illustrate both addition and subtraction as performed in arithmetic.

Thus, in adding 5 to 2 we may begin at 2 and count 5 spaces to the right, obtaining the sum 7. We shall obtain the same result if we begin at 5 and count 2 spaces to the right. This process may be stated in general terms thus :

RULE. To add the number a to the number b, begin at b and count a spaces to the right.

In subtracting 3 from 5 we may begin at 5 and count 3 spaces to the left, thus obtaining 2. This process may be stated as follows :

RULE. To subtract the number a from the number b, begin at b and count a spaces to the left.

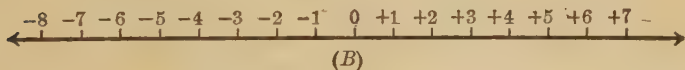
If we attempt to subtract 4 from 3 by the preceding rule, we arrive at the first point of division to the left of zero. Arithmetic has no number to represent such a result ; in fact, the subtraction of 4 from 3 is there regarded as impossible. We can, however, subtract 3 of the 4 units from the 3 units, leaving one of the 4 units unsubtracted. Now in algebra it is both convenient and necessary to speak of subtracting a greater number from a less, and to

call the portion of the greater number, which is unsubtracted, the remainder. The fact that such a subtraction is incomplete is indicated by writing a minus sign before the result; thus, $3 - 4 = -1$. Hence the first point of division to the left of zero may be thought of as corresponding to -1 . Similarly, $3 - 5 = -2$; and to -2 may correspond the second point of division to the left of zero.

In like manner $6 - 9 = -3$, which corresponds to the third point to the left of zero. In the same way the fourth point of division to the left of zero would correspond to -4 , the fifth point to -5 , etc.

Such numbers as -1 , -2 , -3 , etc. are called *negative numbers*. The minus sign is never omitted in writing a negative number, though a letter, as x , may denote one.

The relative order of positive and negative numbers is indicated in the following scale:



ORAL EXERCISES

Perform the following additions and subtractions by counting along the preceding scale:

- | | |
|--------------------|----------------------------|
| 1. Add 4 to 3. | 7. Subtract 4 from 6. |
| 2. Add 3 to + 4. | 8. Subtract 5 from 2. |
| 3. Add 7 to -3 . | 9. Subtract 5 from 3. |
| 4. Add 4 to -4 . | 10. Subtract 5 from -3 . |
| 5. Add 2 to -5 . | 11. Subtract 3 from -4 . |
| 6. Add 6 to -8 . | 12. Subtract 4 from -2 . |

23. Addition of positive and negative numbers. As we have seen, subtraction by the use of scale (*B*) is performed by counting spaces to the left. Now a negative number represents an unperformed subtraction; therefore to add a negative number to another number means to perform this subtraction.

For example, in subtracting 7 from 4, -3 was obtained by beginning at 4 and counting 7 spaces to the left, arriving at 3 to the left. Hence when we wish to add -7 to any number, we count 7 spaces to the left from that number.

To add -9 to 16 we begin at 16 and count 9 spaces to the left, obtaining 7 as the result; that is,

$$+16 + (-9) = +7.$$

Similarly, to add -5 to -3 we begin at -3 and count 5 spaces to the left, obtaining -8 as the result; that is,

$$-3 + (-5) = -8.$$

Hence, in general, to add a negative number n to a given number, begin at the given number and count n spaces to the left.

ORAL EXERCISES

Add the following numbers by the use of scale (*B*):

1. $+5, +2.$ ¹

5. $7, -2.$

9. $4, -6.$

2. $+5, -2.$

6. $-7, -1.$

10. $-3, -4.$

3. $-5, +2.$

7. $-6, -2.$

11. $-5, +5.$

4. $-5, -2.$

8. $6, -3.$

12. $-7, +9.$

The preceding exercises illustrate the correctness of the following working rules:

RULE I. *To find the sum of two or more numbers all of which have the same sign, add the numbers arithmetically, and prefix the common sign to the result.*

/ **RULE II.** To find the sum of two numbers with unlike signs, subtract the number with the smaller numeric value from the number with the greater numeric value, and prefix the sign of the greater number to the result.

These rules are used throughout the whole of algebra. Errors are likely to occur in carrying out processes governed by Rule II. It is important, therefore, that correct habits in adding be formed as rapidly as possible.

* The *algebraic sum* of two or more numbers is the number obtained by adding them according to these rules.

† The *numeric, or absolute, value* of a number is its value regardless of its sign.

Thus, the absolute values of -7 , $+10$, and -15 are 7, 10, and 15 respectively. Note that two different numbers, as $+4$ and -4 , may have the same numeric, or absolute, value.

The algebraic sum of two numbers is not always the same as the sum of their absolute values; for example, the algebraic sum of $+8$ and -3 is $+5$, but the sum of their absolute values is 11.

‡ Hereafter the word *add* will mean *find the algebraic sum*.

ORAL EXERCISES

Perform the indicated additions:

1. $+6 + (+7)$.

7. $-7.1 + 16.3$.

2. $-6 + (-7)$.

8. $8.3 + (-2.6)$.

3. $+6 + (-7)$.

9. $-5.3 + 8.7$.

4. $-6 + (+7)$.

10. $-3 + (+3) + (+5)$.

5. $+8 + (-6)$.

11. $2 + (+6) + (+4) + (-3)$.

6. $-8 + (+6)$.

12. $7 + (-2) + (-3) + (+5)$.

Answer the questions asked in the following:

$$13. 8 + ? = 10.$$

$$18. -9 + ? = -6.$$

$$14. 8 + ? = 2.$$

$$19. -10 + ? = -15.$$

$$15. 9 + ? = 12.$$

$$20. -10 + ? = 4.$$

$$16. 9 + ? = 5.$$

$$21. 12 + ? = 6.$$

$$17. -9 + ? = -10.$$

$$22. -12 + ? = 6.$$

24. Subtraction of positive and negative numbers. If we wish to subtract 9 from 14, we may do so by answering the question, "What number added to 9 gives 14?" By answering a similar question we can subtract 7 from 13, or 16 from 27, or any number a from another number b . Exercises 13-22, above, are therefore exercises in subtraction, for each asks a question similar to the one in the first sentence of this paragraph.

This point of view brings out the relation that the operation of subtraction bears to that of addition.

ORAL EXERCISES

Perform the following subtractions by answering in each case the question, "What number added to the first number gives the second number?"

Subtract:

$$1. 4 \text{ from } 6.$$

$$6. -4 \text{ from } 4.$$

$$2. 7 \text{ from } 12.$$

$$7. 6 \text{ from } -11.$$

$$3. 8 \text{ from } 4.$$

$$8. -7 \text{ from } -3.$$

$$4. 12 \text{ from } 5.$$

$$9. 10 \text{ from } -17.$$

$$5. -7 \text{ from } 8.$$

$$10. -25 \text{ from } 12.$$

11. In Exercises 1-10, change the sign of the first number (if +, to -; if -, to +) and then add it to the second number. Are the same answers obtained as before?

These results illustrate the following principles:

I. *Subtracting a positive number is the same in effect as adding a negative number of the same absolute value.*

To illustrate: a decrease of \$100 in a man's assets is equivalent to an increase of \$100 in his liabilities, provided we consider his financial standing as a whole in each case.

II. *Subtracting a negative number is the same in effect as adding a positive number of the same absolute value.*

To illustrate: a decrease of \$75 in a man's liabilities is equivalent to an increase of \$75 in his assets, as far as his net financial condition is concerned.

Hence, for the subtraction of positive and negative numbers, we have the

RULE. *Change the sign of the subtrahend (if +, to -; if -, to +). Then find the algebraic sum of the subtrahend (with its sign changed) and the minuend.*

ORAL EXERCISES

Subtract the second number (subtrahend) from the first (minuend) in Exercises 1-18:

1. $8, + 3.$

7. $- 14, + 5.$

13. $- 14, - 17.$

2. $8, - 3.$

8. $- 14, - 5.$

14. $0, + 2.$

3. $+ 3, - 8.$

9. $+ 7, + 7.$

15. $- 2, - 2.$

4. $+ 3, + 8.$

10. $- 7, - 7.$

16. $2, - 2.$

5. $14, + 5.$

11. $7, - 7.$

17. $- 3.5, 2.2.$

6. $14, - 5.$

12. $- 7, + 7.$

18. $6.5, - 1.7.$

Supply the missing numbers in the following :

19. $13 - (+2) - (+4) = ?$

29. $-8 + ? = -7.$

20. $-11 - (+2) - (+4) = ?$

30. $-8 + ? = 8.$

21. $-10 - (-2) - (-4) = ?$

31. $9 - ? = 4.$

22. $18 - (-4) - (+6) = ?$

32. $-6 - ? = -4.$

23. $+7 + ? = 11.$

33. $+5 - ? = -8.$

24. $-7 + ? = -11.$

34. $-8 - ? = 5.$

25. $-4 + ? = 0.$

35. $-4 - ? = 0.$

26. $+5 + ? = 0.$

36. $3 - ? = 0.$

27. $+5 + ? = 5.$

37. $4 - ? = 15.$

28. $-7 + ? = -4.$

38. $7 + ? = -3.$

PROBLEMS

1. A thermometer registers -10° . What is the temperature when the thermometer registers twice as far below zero? three times? four times? five times?

2. One man owes \$5. A second man owes twice as much. How much does the second man owe?

3. The car shortage for March on a certain line was 376. This is the same as saying the line had -376 cars more than were needed. What number would express a shortage twice as great? three times as great?

4. Death Valley is about 325 feet below sea level; that is, its elevation is -325 feet. The Dead Sea is four times as far below sea level as Death Valley. What number would express the elevation of the Dead Sea?

25. **Multiplication of positive and negative numbers.** From the preceding problems we see that even in arithmetic we might multiply negative numbers as well as positive

numbers. This is more often necessary in algebra than in arithmetic, as algebra deals with negative numbers as well as with positive numbers.

Four cases of the multiplication of numbers may arise; for example,

$$(+3) \cdot (+2) = ? \quad (+3) \cdot (-2) = ?$$

$$(-3) \cdot (+2) = ? \quad (-3) \cdot (-2) = ?$$

From arithmetic we have learned that $(+3) \cdot (+2) = (+3) + (+3) = +6$.

In $(-3) \cdot (+2)$, -3 is to be added twice, which is the same as adding -6 once; hence $(-3) \cdot (+2) = -6$; in $(+3) \cdot (-2)$, $+3$ is to be subtracted twice, which is the same as subtracting $+6$ once. Therefore $(+3) \cdot (-2) = -6$. Lastly, $(-3) \cdot (-2)$ means that -3 is to be subtracted twice, which is the same as subtracting -6 once. But subtracting -6 is the same as adding $+6$. Therefore

$$(-3) \cdot (-2) = +6.$$

In general terms,

$$(+a) \cdot (+c) = +ac,$$

$$(-a) \cdot (+c) = -ac,$$

$$(+a) \cdot (-c) = -ac,$$

$$(-a) \cdot (-c) = +ac.$$

Therefore we have the

RULE. *The product of two numbers having like signs is a positive number, and the product of two numbers having unlike signs is a negative number.*

ORAL EXERCISES

Find the products of the following :

- | | | |
|-----------------|----------------------|----------------------|
| 1. $+ 3, + 4.$ | 8. $+ 8, - 4.$ | 15. $- 4, - 6, - 2.$ |
| 2. $+ 4, + 11.$ | 9. $- 7, - 8.$ | 16. $12, + 0, - 4.$ |
| 3. $- 4, + 6.$ | 10. $- 1.6, - .2.$ | 17. $8, - 10, - 0.$ |
| 4. $- 2.5, 3.$ | 11. $+ 0, + 5.$ | 18. $- 4, + 8, - 8.$ |
| 5. $- 7, + 9.$ | 12. $- 8, 0.$ | 19. $- 3, - 4, - 5.$ |
| 6. $- 7, - 5.$ | 13. $+ 4, - 8, + 6.$ | 20. $3, - 3, + 9.$ |
| 7. $- 11, + 9.$ | 14. $+ 4, - 7, - 6.$ | 21. $- 1.5, 0.4.$ |

NOTE. The famous German mathematician Leopold Kronecker (1823-1891) once observed that "the good Lord made the positive integers, but man is responsible for all the rest of the numbers." This expresses the truth about numbers as accurately as one can in a single sentence. We count objects from our earliest years, and so use the positive integers naturally. It is only when we come to study mathematics that the necessity for any other kind of numbers is forced upon us. Here we see that negative numbers are a great convenience if we wish to represent the relations between objects where oppositeness in any of its many forms is involved. But the artificial character of negative numbers delayed their intelligent use for many hundred years. To be sure, the Hindus said that "the square of negative is positive," but the statement probably did not mean anything to those who read it. It was not until after the time of Descartes (see page 284) that the rules for operating on negative numbers were understood, even by great mathematicians.

26. Division of positive and negative numbers. When 16 is divided by 8, the result is 2. To test whether 2 is the correct value of $16 \div 8$, we check thus: $8 \cdot 2 = 16$. To check $12 \div 4 = 3$, we multiply the quotient 3 by the divisor 4 and get 12, the number divided. This test will be used to determine whether when positive and negative numbers

are divided, the answer is positive or negative. All the cases which may arise are represented by the four questions :

$$(a) + 16 \div (+ 8) = ?$$

$$(c) + 16 \div (- 8) = ?$$

$$(b) - 16 \div (+ 8) = ?$$

$$(d) - 16 \div (- 8) = ?$$

These questions are answered thus :

$$(a) + 16 \div (+ 8) = + 2, \text{ because } + 2 \cdot (+ 8) = + 16.$$

$$(b) - 16 \div (+ 8) = - 2, \text{ because } - 2 \cdot (+ 8) = - 16.$$

$$(c) + 16 \div (- 8) = - 2, \text{ because } - 2 \cdot (- 8) = + 16.$$

$$(d) - 16 \div (- 8) = + 2, \text{ because } + 2 \cdot (- 8) = - 16.$$

In (a) and (d) the dividends and the divisors have like signs and the quotients are positive. In (b) and (c) the dividends and the divisors have unlike signs, and the quotients are negative.

Therefore we have the

RULE. *When two numbers having like signs are divided, the quotient is a positive number; when two numbers having unlike signs are divided, the quotient is a negative number.*

27. Division by zero. The result of multiplication by zero is given a definite meaning in arithmetic and algebra, namely zero; but in both subjects division by zero is always excluded. If zero were used as a divisor, numerous contradictions would arise, of which the following is an illustration :

$$\text{Obviously,} \quad 0 \cdot 3 = 0,$$

$$\text{and} \quad 0 \cdot 5 = 0.$$

$$\text{Therefore} \quad 0 \cdot 3 = 0 \cdot 5.$$

Dividing each by zero, $3 = 5$,
which is false.

NOTE. The Hindus were the first to express the laws that govern the operations with the number 0. In fact, they were the first to have such a symbol. In the twelfth century a Hindu writer stated that $a + 0 = a$, that $\sqrt{0} = 0$, and that $0^2 = 0$. Of course he did not express himself in terms of these symbols, but in the notation of his time and country.

ORAL EXERCISES

In Exercises 1-12 divide the first number by the second :

- | | | |
|------------------|------------------|-------------------|
| 1. $+ 10, + 5$. | 5. $- 21, - 3$. | 9. $- 12, + 3$. |
| 2. $- 10, - 2$. | 6. $- 6, + 6$. | 10. $+ 12, - 3$. |
| 3. $- 15, + 5$. | 7. $0, + 4$. | 11. $- 12, - 3$. |
| 4. $+ 14, - 7$. | 8. $- 4, - 4$. | 12. $+ 36, 2$. |

MISCELLANEOUS ORAL EXERCISES

Simplify the following :

- | | | |
|-------------------------|----------------------------|---------------------------|
| 1. $(6) + (5)$. | 11. $- 6 + 9$. | 21. $8(-4)$. |
| 2. $(6) - (5)$. | 12. $9 + (-11)$. | 22. $(-5)(-11)$. |
| 3. $(6) - (-5)$. | 13. $12 - 16$. | 23. $(-3)8$. |
| 4. $(-6) + (5)$. | 14. $- 16 - 12$. | 24. $- 4 \cdot 6$. |
| 5. $(-6) - (-5)$. | 15. $18 - 16$. | 25. $6 \cdot 0$. |
| 6. $- 6 + 5$. | 16. $+ 8 - 0$. | 26. $0 \cdot (-8)$. |
| 7. $(-8) - (4)$. | 17. $0 - 2$. | 27. $5 \cdot 8$. |
| 8. $- 8 - 4$. | 18. $(-3)(5)$. | 28. $- 3 \cdot 7$. |
| 9. $- 11 + (-12)$. | 19. $(-3)(7)$. | 29. $12 \div (-3)$. |
| 10. $- 6 - (-9)$. | 20. $(-7)5$. | 30. $- 6.8 \div (-0.2)$. |
| 31. $- 38 \div (-2)$. | 35. $0 \div 4 \div (-5)$. | |
| 32. $45 \div (+15)$. | 36. $8.4 \div .4$. | |
| 33. $- 45 \div (+15)$. | 37. $3 - 7 + 8$. | |
| 34. $0 \div (-7)$. | 38. $- 4 + 5 - 2 + 1$. | |

Supply the missing term in the following:

$$39. \frac{-20}{?} = 5.$$

$$40. \frac{-12}{?} = 2.$$

$$41. \frac{+36}{?} = -4.$$

Add:

$$\begin{array}{r} 42. \quad 8 \\ -2 \\ 3 \\ -6 \\ \hline \end{array}$$

$$\begin{array}{r} 43. \quad 6 \\ -2 \\ +3 \\ -4 \\ \hline \end{array}$$

$$\begin{array}{r} 44. \quad 8 \\ -6 \\ -2 \\ 5 \\ \hline \end{array}$$

$$\begin{array}{r} 45. \quad -4 \\ +9 \\ +3 \\ -6 \\ \hline \end{array}$$

$$\begin{array}{r} 46. \quad 8 \\ -7 \\ -5 \\ +4 \\ \hline \end{array}$$

Simplify:

$$47. 3 \cdot 8 \div 2.$$

$$59. -2(3)^2(2).$$

$$48. -3(8) \div (-2).$$

$$60. -3(-2)(-3)^3.$$

$$49. 3(-8) \div 2.$$

$$61. -2(-4)^2(-5).$$

$$50. 4 \cdot 6(-7) \div (-16).$$

$$62. 2(4)^2 \cdot 5.$$

$$51. 18 \div (3) \cdot 6 \div (-4).$$

$$63. 3(5)^2 \cdot (-6).$$

$$52. 2^2.$$

$$64. -4(5)^2 \div 2.$$

$$53. (-2)^2.$$

$$65. -4(5)^2 \div (-2).$$

$$54. (3)^3.$$

$$66. -4(6)^2 \div 6.$$

$$55. (-3)^3.$$

$$67. 4(-6)^2 \div (-6).$$

$$56. (-3)^3 + (3)^3.$$

$$68. 3 \cdot 5(-7)^2 \div 21.$$

$$57. -(-4)^2.$$

$$69. 3 \cdot 5(7)^2 \div (-21).$$

$$58. -(-4)^3.$$

$$70. -4 \cdot 2 \cdot (9)^2 \div (6)^2$$

REVIEW EXERCISES

Simplify the following:

$$1. 4^2 - (2)^2.$$

$$5. (5 - 2)(3 + 2).$$

$$2. 4^2 + (-2)^2.$$

$$6. (5 - 3)(4 + 2).$$

$$3. 3^3 - (-2)^2.$$

$$7. (5 - 2)(7 - 3) \div (3 - 9).$$

$$4. 3^3 + (-2)^3.$$

$$8. (-1)^2 + (-1)^3 - (-2)^2 - (-2)^3.$$

$$9. 9 + 3 \cdot 2 - 18 \div 3.$$

$$10. 5^2 + 4 \div 2 \div (-6) \cdot (-3).$$

$$11. 3 \cdot 6 \div 9 + 2 \cdot 6 \div 4 - (-3)^2.$$

$$12. (-3)^2 + 2 \cdot 3 \cdot (-4) \div 4^2.$$

$$13. (+2)^2 - 2(2)(-3)^2 + (-3)^3.$$

$$14. 3(4)^2(-5) - (-5)^2.$$

$$15. 4^3 - 3(4)^2(-3) + 3(4)(-3)^2 - (-3)^3.$$

$$16. 3^3 + 3(3)^2(-2)^2 + 3(3)(-2)^3 + (-2)^4.$$

$$17. (-3)^3 - 3(-3)^2(0) + 3(-3)(0)^2 - 0^3.$$

$$18. (-5)^2 + 3(-5)^2(+4) + 3(-5)(+4)^2 - (4)^3.$$

$$19. 2(-3)^3 + 6(-5)^2(+4) + 6(-5)(+4)^2 + (4)^3.$$

If $x = -3$, $y = 2$, find the value of the following:

$$20. y^2. \quad 24. y^4. \quad 28. 3y^3. \quad 32. x^2 + y^2.$$

$$21. y^3. \quad 25. x^4. \quad 29. 3x^3. \quad 33. x^3 + y^3.$$

$$22. x^2. \quad 26. 3y^2. \quad 30. 5xy. \quad 34. 2x^2 + 4xy + y^2.$$

$$23. x^3. \quad 27. 3x^2. \quad 31. 5x^2y^2. \quad 35. 2x^2 - 4xy + y^2.$$

$$36. x^3 - 3x^2y + 3xy^2 - y^3. \quad 39. 3x^2 - 6xy + 5y^2.$$

$$37. x^4 - y^4. \quad 40. (x + y)^2(x - y)^2.$$

$$38. 3x^2 + 6xy + 3y^2. \quad 41. 2x^3 + 6x^2y + 6xy^2 + 2y^3.$$

$$42. x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4.$$

$$43. \text{ Does } 4x - 2 = 2x - 12, \text{ if } x = -5?$$

$$44. \text{ Does } 3x - 5 = 2x + 8, \text{ if } x = 12?$$

$$45. \text{ Does } x^2 - x - 12 = 0, \text{ if } x = 3? \text{ if } x = -3? \text{ if } x = +4?$$

$$46. \text{ Does } x^2 + 10x = -25, \text{ if } x = \frac{1}{2}? \text{ if } x = -4? \text{ if } x = -5?$$

$$47. \text{ Does } 2x^2 - 20x + 50 = 0, \text{ if } x = -5? \text{ if } x = 5? \text{ if } x = \frac{1}{3}?$$

$$48. \text{ Does } x^2 + x - 12 = 0, \text{ if } x = +4? \text{ if } x = -8? \text{ if } x = -4? \text{ if } x = 3?$$

$$49. \text{ Does } x^2 - 7x = -12, \text{ if } x = 6? \text{ if } x = \frac{1}{2}? \text{ if } x = -3?$$

50. Using positive numbers to represent excess, and negative numbers to represent shortage, express the following records on the number of freight cars available during the first six months of a certain year: January, excess of 560 cars; February, excess of 420 cars; March, shortage of 176 cars; April, shortage of 360 cars; May, shortage of 574 cars; June, excess of 564 cars.

51. The temperature is $+15^{\circ}$. What will represent the temperature after a drop of (a) 5° ? (b) 10° ? (c) 20° ?

52. The temperature is now -18° . What will it be (a) after a rise of 8° ? (b) after a rise of 14° ? (c) after a rise of 30° ? (d) after a change of -10° ? (e) after a change of $+30^{\circ}$?

53. The temperature at 6 A.M. was -12° . During the day it rose at the rate of 3° per hour. What was the temperature at 9 A.M.? at 10 A.M.? at 12 M.?

54. The speed of a trolley car going in a certain direction, taken at half-hour intervals, was as follows: 20 mph (miles per hour), 25 mph, 15 mph, 0, 10 mph, 0, -25 mph, -10 mph, 0, -5 mph. Plot these values, using the line graph. What is the meaning of the negative velocities given?

Where an initial direction has been assumed, we are frequently confronted with negative directions. This is particularly true in physics and the applied sciences.

55. A man withdraws \$50 from his bank one week. The second week he withdraws \$10 less than the first week; the third week \$10 less than the second week; and so on for a period of eight weeks. Represent these withdrawals by means of a line graph. Interpret the negative values found after the sixth week.

CHAPTER IV

ADDITION

28. Monomials. A *monomial*, or *term*, is a number symbol which is not the indicated sum or difference of two or more number symbols.

Thus 6 , $-b$, a^2 , b^2c , and $-3cd^2$ are monomials, or terms.

Frequently, where no confusion would arise, expressions like $(5-3)$, $2(x+y)$, $4\sqrt{x^3}$, and $\sqrt{b-x}$ are called terms, for in such cases one thinks not of the parts but of a single number for which the whole stands. We may think of the expression $(5-3)$ not as two numbers but as the one number 2 .

29. Similar terms. Terms are *similar* when they are alike in all respects except their coefficients.

Thus 2 , -6 , and 8 are similar terms, as are b , $3b$, and $-5b$. Also $\sqrt{5}$ and $2\sqrt{5}$ are similar terms, as are $2a^2x$, $-5a^2x$, and $6a^2x$.

30. Addition of similar terms. It is clear that $6b + 4b = 10b$; also, that $3ax + 2ax + 8ax = 13ax$. In like manner $4abc + (-3abc) + 12abc + (-8abc) = 5abc$, and the sum of $3xy$, $-xy$, $8xy$, $-4xy$, and xy is $7xy$. The terms $-xy$ and $+xy$ are equivalent to $-1xy$ and $+1xy$.

For adding similar terms we have the

RULE. Find the algebraic sum of the numeric coefficients and prefix this result to the common literal part.

ORAL EXERCISES

Find the algebraic sum of :

1. $5b, 3b.$

6. $+5r, -8r.$

2. $-12a, +7a.$

7. $-11a, +19a.$

3. $-9c, -3c.$

8. $-2ac, -3ac.$

4. $-2a, -3a.$

9. $-xy, +3xy, -4xy.$

5. $+8x, -2x.$

10. $-6mn, +2mn, -mn.$

Combine :

11. $ax + 2ax - 8ax + 4ax.$

12. $3am - 5am - 9am + 7am.$

13. $5ab - 4ab + 11ab - 8ab.$

14. $9ab - 7ab + 4ab - 5ab.$

15. $4axy + 3axy - 7axy + 14axy.$

16. $-abc + 4abc + abc - 5abc + 11abc.$

17. $10bc - 6bc + 12bc - 8bc.$

18. $4bc - 7bc + 10bc - 8bc + 12bc.$

19. $6xy, -xy, -7xy, +xy, 8xy, -4xy, xy.$

20. $3(x+y), -4(x+y), +6(x+y), -(x+y), -5(x+y).$

31. Order in adding terms. Obviously, $3+4+6=4+6+3=6+4+3$, etc. This illustrates the law that in addition the terms may be arranged and added in any order.

Hence $5b+6c=6c+5b$, and the sum of 4 and b is either $4+b$ or $b+4$; also $b+c=c+b$ and $x+y+z=z+y+x$.

32. Addition of dissimilar terms. An algebraic expression for the sum of two terms which are not similar, such as $5x$ and $3k$, is obtained by writing them one after another with the plus sign between them; thus, $5x+3k$.

Similarly, the sum of $4a$ and $-5m$ is $4a + (-5m)$ or $4a - 5m$, and the sum of $2x$, $4y$, and $-2z$ is $2x + (+4y) + (-2z)$, or, omitting the parentheses, $2x + 4y - 2z$.

For adding dissimilar terms we have then the

RULE. Write the terms one after the other in any order, giving to each its proper sign.

EXERCISES

Write the sum of:

1. $2a$, $3b$, $-2c$.
2. $3x$, $-b$, $5y$, 8 .
3. $2a^2b$, $3bx$, $-2cy$, ab^2 .
4. $6x^3y$, $-4xy^3$, $2c^3y$, $-cy^3$.
5. $5x$, $-2a$, $3b$, $-4x$, $4y$.
6. $2a^2$, $+3b^2$, $-8c^2$, $-5b^2$, $6a^2$.
7. $4a$, $-7b$, $+4c$, $6b$, $-3c^2$.
8. $4a^3b$, $-4ab^3$, $-4a^2b$, $4ab^2$, $3a^3b$, $3ab^3$.
9. $7a^2bc$, $-5abc^2$, $9ab^2c$, $7abc^2$, $-3a^2bc$, $-6ab^2c$.
10. $10a^3b$, $3ab^3$, $-4a^2b$, $-7ab^3$, $6a^2b^3$, $-3a^2b$.

Simplify:

11. $5 - 9 - 2 + 15 - 20 + 10$.
12. $14ac - 8xy + 4ac - 3xy - 7ac + 12xy$.
13. $8x + 7a - 14a - 9x + 4x$.
14. $12y - 18y + 9b + 25y - 24b$.
15. $4z^2 - 5y^2 - 10y^2 + z^2 - 8y^2$.
16. $5ab - 7xy - 11ab + 14ab - xy - 7ab$.
17. $-4ab^2 + 6xy^2 - 13ab^2 + 4xy^2 + ab^2 - 0xy^2$.
18. $7a^3 + 4a^2b - 6ab^2 + a^3 + 5b^3 + 5a^2b - 2b^3 + 3ab^2$.
19. $7b^2d - 5ab^2 + 6b^2d - ab^2 + 9ab^2 - 10b^2d$.
20. $-b^4 - 18a^2 + 19b^4 - 0b^4 + 13a^2 + b^4 - 0a^2$.
21. $10ab^3 + 4cd^2 - 0ab^3 + 14ab^3 - 11ab^3 - 23cd^2 + 0cd^2$.

$$22. 11\sqrt{b} - 15\sqrt{b} + 20\sqrt{b} - 0\sqrt{b}.$$

$$23. 4\sqrt{a-b} - 2\sqrt{a-b} + 7\sqrt{a-b} - 6\sqrt{a-b}.$$

$$24. 5(2a-b) + 2(2a-b) - 4(2a-b) - (2a-b).$$

$$25. 4\sqrt{a^2b} + 3\sqrt{a^2b} - 7\sqrt{a^2b} + 4\sqrt{a^2b} - 9\sqrt{a^2b}.$$

33. Addition of polynomials. A *polynomial* is an algebraic expression consisting of two or more terms.

An expression is not called a polynomial if any of its terms contain a letter under the radical sign. Thus $\sqrt{a-2} + 3$ is not called a polynomial.

A *binomial* is a polynomial of two terms.

Thus $x + 3$, $3x - 5$, $h^2 - 9$, and $3 - 2x^2$ are binomials.

A *trinomial* is a polynomial of three terms.

Thus $x - y - 3$, $2x + y - 8$, $3x + 4y - 7z$, $m - n - p$, and $2x + 6r - 5s$ are trinomials.

EXAMPLE

Add the following polynomials: $2a - 3b - ac^2$, $2b + 3ac^2$, $-3a - 7ac^2 + 11$, and $4a + 3b - 5$.

$$\begin{array}{r} \text{Solution.} \quad + 2a - 3b - ac^2 \\ \quad \quad \quad + 2b + 3ac^2 \\ \quad \quad \quad - 3a \quad \quad - 7ac^2 + 11 \\ \quad \quad \quad + 4a + 3b \quad \quad - 5 \\ \hline \text{Sum,} \quad \quad 3a + 2b - 5ac^2 + 6 \end{array}$$

For the addition of polynomials we have the

RULE. Write similar terms in the same column.

Find the algebraic sum of the terms in each column and write the results in succession with their proper signs.

34. Checks. A *check* on an operation is a second operation which tests the correctness of the first.

For example, in arithmetic the result of subtraction is checked by addition; thus, the check for $10 - 4 = 6$ is $6 + 4 = 10$. The check for the result of division is multiplication. Thus the check for $144 \div 8 = 18$ is $18 \cdot 8 = 144$.

To check addition we sometimes add columns in the opposite direction. However, there is really no check for addition that is absolutely certain to detect errors. If the numbers in the column are not all of the same sign, the result can be partially checked by adding the positive and negative numbers separately and finding the algebraic sum of the two results.

Perhaps the most common check for the addition of polynomials is to substitute some number or numbers for the letters in the expression. Each term which is being added will then have some numeric value. The numeric value of the algebraic sum of the polynomials should be equal to the algebraic sum of the numeric values of the individual polynomials. If this does not turn out to be the case, some error in the original addition has been made.

Thus, in the example given above, let $a = 1$, $b = 2$, and $c = 3$. The work of addition and the corresponding numeric check may then be represented as follows:

$$\begin{array}{rcl}
 2a - 3b - ac^2 & (= & 2 - 6 - 9 \quad = -13) \\
 + 2b + 3ac^2 & (= & +4 + 27 \quad = +31) \\
 - 3a & - 7ac^2 + 11 & (= -3 \quad - 63 + 11 = -55) \\
 4a + 3b & - 5 & (= 4 + 6 \quad - 5 = +5) \\
 \hline
 3a + 2b - 5ac^2 + 6 & (= & 3 + 4 - 45 + 6 = -32)
 \end{array}$$

The sum of the numeric values of the several expressions is in this case the same as the numeric value of the sum of the expressions, namely -32 , thus indicating, although not proving, that the addition has been correctly done.

EXERCISES

Add the following polynomials and check the results:

$$\begin{array}{r} 1. \quad x + 2 \\ 3x - 4 \\ \hline 4x - 6 \end{array}$$

$$\begin{array}{r} 2. \quad x - z - 2 \\ 2x + z - 7 \\ \hline 3x - z - 9 \end{array}$$

$$\begin{array}{r} 3. \quad x - y + z \\ 2x - 3y + 3z \\ \hline 3x - 2y + 4z \end{array}$$

4. $4x + 4y$, $5x - 9y + 4z$, and $4x - 4y - 4z$.

5. $8x - 7y + 4z$, $6x - 5z$, and $3x + 7y - 6z$.

6. $x^2 - x - 2$, $2x^2 - 4x + 6$, and $4x^2 + 7x - 9$.

7. $x^2 - 5x + 2$, $2x - 4x^2 + 8$, and $11x^2 - 10x - 18$.

8. $3d - 4d^2 - 6$, $d^2 + d - 1$, $4d - 3d^2 + 3$, and $4 - d - 5d^2$.

9. $4a - 5b + 6c$, $7b - 2c$, and $6a - 4b - 3c$.

10. $a + 2b + 3c$, $b - 3c + a$, and $c - 2a - 4b$.

11. $6x - 7y + 8z$, $2y - 12z + x$, and $10z - 4y$.

12. $4a - 3b - 3c$, $3c - 2a - b$, and $4b + 8c$.

13. $10bc - ac$, $7ab - 3bc$, and $-12bc - ab$.

14. $2a^2 - 4a + 5$, $5a - 3a^2 + 2$, $6a - 4a^2$.

15. $3 - a^2 + 3a$, $-6 + 3a^2 - 2a$, $6a^2 - 4a$.

16. $a - x - 4c$, $5 - 6x + 5a$, $6c - 3a + 9$, and $a - 2c - 2x - 12$.

17. $a + c + x - 5$, $3a + 2x + 9 - 2c$, $a - 7 - 2c - x$, and $6 - 3x - 4a + c$.

18. $x - 2 - 4c$, $x - a - 3c$, $a - 4c - 6$, and $3x - 7 + 12a$.

19. $3x + 4y + 6z$, $2x - 2y + 4z$, and $4x - 3y + 11z$.

20. $2x - 10z$, $4x + 3y$, $5z - y$, and $3x - 4y + z$.

21. $2 - 4(x - y) + 2(x + z)$, $5 - 10a + 3(x - y)$, $-2(x + z) + 7$, and $5(x - y) + (x + z) + 3a$.

Copy one & check

22. $a + c - 2(b - d)$, $7(b - d) + 6c - 11a$, and $10a - 4(b - d)$.

23. $a + x + 3(y + z)$, $2a - 3x + 4b$, $7x - 10(y + z)$,
and $7b - 3a + z + 2(y + z)$.

Combine similar terms in the following polynomials:

24. $6a^2 - 12b^2 + 11c^2 + 14b^2 - 2a^2 - c^2 + 3b^2 + 4c^2$.

25. $a^2 + 2ab + b^2 - 3ab - 4a^2 - b^2 + 8a^2 - 4ab + b^2$.

26. $5bc - b^2 + c^2 + 4bc - 3c^2 + 5b^2 - 4bc - 5c^2$.

27. $5w^2 - 6w + 11 - 4w - 8 - 3w^2 + 7w - 18 + 13w$.

28. $4y^2z - yz + z^2 - 3yz + 4yz^2 - 2y^2z + 4z^2 + 3yz^2 + y^2z$.

29. $\frac{1}{2}b + \frac{1}{3}c - \frac{1}{4}d + 5$, $\frac{1}{2}c - \frac{2}{3}b - c$, $\frac{1}{2}d + \frac{3}{4}b + 7$.

The sum of $2a$ and $3a$ may be written $(2 + 3)a$. This is not usually done, as the sum of $2a$ and $3a$ can be written $5a$. In adding ax and $3x$ the a and 3 cannot be combined. The sum $ax + 3x$ can be written $(a + 3)x$. Similarly, $bx - 5x = (b - 5)x$; $bx + x = (b + 1)x$; and $2x + ax + cx = (2 + a + c)x$; also $a(y + z) + c(y + z) = (a + c)(y + z)$.

Express as one term so that x will have a binomial or a polynomial coefficient:

30. $ax + 4x$.

35. $4ax - 4cx - x$.

31. $2ax + bx$.

36. $2ax - 2bx - x + b^2x$.

32. $4bx - x$.

37. $cx - 5bx - x + 3cx$.

33. $ax + 2bx + cx$.

38. $2x - 3ax + bx - cx$.

34. $ax - 3x + bx$.

39. $.3x + 2.5ax - .7bx + .5cx$.

CHAPTER V

SIMPLE EQUATIONS

Memoranda
35. Definitions. An equation has already been defined as a statement of equality between two equal numbers or number symbols.

Thus $3 = 7 - 4$, $x - 2y = 3x + y - 2x - 3y$, $4a = 2 + 12$, and $x^2 - x - 6 = 0$ are equations.

Memoranda
The part of the equation on the left of the equality sign is called the *first* or *left member*; that on the right, the *second* or *right member*.

In an equation the letter whose value is sought is called the *unknown letter* or the *unknown*.

Thus, in the equation $2x = 4$, x is the unknown, and in the equation $3h - 5 = 13$, h is the unknown.

The process of finding the value of the unknown letter in an equation is called *solving the equation*. If the value of the unknown be substituted for it in the equation and if, when the result is simplified, the left member of the equation becomes identical with the right, then the unknown is said to *satisfy* the equation.

The presence of an equality sign between two algebraic expressions is not sufficient to form an equation.

Thus $x = x + 1$ is not an equation, since there is no number which equals itself increased by 1.

ORAL EXERCISES

Does the number in parenthesis at the right of each equation satisfy that equation?

1. $x - 3 = 2.$ (5) 7. $2y + 5 = 11.$ (3)

2. $4 - y = 2.$ (3) 8. $6y - 10 = 2.$ (1)

3. $3 + t = 6.$ (3) 9. $4 - 5y = -1.$ (6)

4. $4 - m = 2.$ (2) 10. $2m - 2 = 0.$ (1)

5. $2z + 3 = 5.$ (1) 11. $4x - 5 = 7.$ (3)

6. $5 - 3x = 5.$ (1) 12. $12s + 5 = 29.$ (1)

36. Axioms. An *axiom* is a statement whose truth is accepted without proof.

In solving equations, constant use is made of the following axioms:

AXIOM I. *If the same number is added to each member of an equation, the result is an equation.*

Thus, adding 4 to each member of $n - 4 = 7$ gives $n = 11$. In this case the use of Axiom I gives us the solution to the original equation.

ORAL EXERCISES

Find the value of the unknown in:

1. $x - 10 = 7.$ 6. $s - 1 = 3.$ 11. $-6 + c = 7.$

2. $n - 3 = 5.$ 7. $4 + 10 + r = 3.$ 12. $-5 + d = 3.$

3. $d - 4 = 4.$ 8. $-4 + m = 1.$ 13. $0 = x - 2.$

4. $x - 3 = 6.$ 9. $n - 2 = 0.$ 14. $h - 5 = 0.$

5. $n - 2 = 13.$ 10. $x - 8 = 0.$ 15. $0 = -3 + x.$

AXIOM II. *If the same number is subtracted from each member of an equation, the result is an equation.*

Thus, subtracting 5 from each member of $n + 5 = 10$ gives $n = 5$. Axiom II states that if this subtraction is performed, the resulting statement is true.

$$n - 5 = 5$$

$$n = 10$$

ORAL EXERCISES

Find the value of the unknown in :

- | | | |
|--------------------|---------------------|---------------------|
| 1. $x + 5 = 8$. | 8. $3 = x + 2$. | 15. $y + 3 = 16$. |
| 2. $a + 3 = 7$. | 9. $9 = a + 4$. | 16. $3 + x = 5$. |
| 3. $r + 15 = 21$. | 10. $12 = d + 5$. | 17. $2 + t = -8$. |
| 4. $5 + d = 20$. | 11. $12 + x = 15$. | 18. $3 + v = 9$. |
| 5. $2 + m = 20$. | 12. $x + 2 = 10$. | 19. $12 + x = 2$. |
| 6. $h + 5 = 7$. | 13. $2 + y = -3$. | 20. $x + 4 = 1$. |
| 7. $10 + b = 16$. | 14. $m + 10 = 16$. | 21. $z + 11 = 20$. |

AXIOM III. *If each member of an equation is multiplied by the same number, the result is an equation.*

When an equation is in the form $\frac{n}{3} = 5$, it can be solved by multiplying each member by 3, giving $n = 15$. Axiom III states that if this multiplication is performed, the result is true.

ORAL EXERCISES

Find the value of the unknown in :

- | | | | | |
|------------------------|------------------------|----------------------------------|----------------------------------|------------------------------------|
| 1. $\frac{x}{3} = 1$. | 3. $\frac{a}{5} = 8$. | 5. $\frac{n}{4} = 10$. | 7. $\frac{k}{6} = \frac{1}{3}$. | 9. $\frac{1}{4} = \frac{l}{8}$. |
| 2. $\frac{x}{2} = 4$. | 4. $\frac{x}{2} = 6$. | 6. $\frac{n}{4} = \frac{1}{2}$. | 8. $\frac{r}{4} = -3$. | 10. $\frac{1}{5} = \frac{m}{10}$. |

AXIOM IV. *If each member of an equation is divided by the same number (not zero), the result is an equation.*

In the equation $4n = 12$, divide both members by the coefficient of n . Thus, dividing both members by 4 we get $n = 3$. Axiom IV states that if this division is performed, the result is an equation.

ORAL EXERCISES

Find the value of the unknown in :

1. $6a = 12$.

5. $3k = 15$.

9. $56 = 14n$.

2. $8n = 24$.

6. $2m = 6$.

10. $18 = 6r$.

3. $7d = 28$.

7. $12x = 60$.

11. $125 = 25x$.

4. $5n = 30$.

8. $24 = 8m$.

12. $13y = 65$.

State the axiom or axioms which apply to the solution of each of the following equations :

13. $3n - 7 = 2$.

17. $\frac{3c}{2} = 6$.

21. $8h + 2 = 10$.

14. $2x + 2 = 4$.

18. $8h = 16$.

22. $\frac{2r}{5} = 4$.

15. $5r - 3 = 12$.

19. $7d + 7 = 21$.

23. $6l = 18$.

16. $7s - 2 = 12$.

20. $3x - 4 = 5$.

24. $\frac{4x}{3} = 4$.

In changing the form of the equation, by the application of an axiom, we do not change the value of the unknown. We merely discover the value it had all the time. These axioms are used almost constantly in the solution of all kinds of equations, whether they involve numbers, letters standing for known numbers, unknown numbers, or all of these together.

EXAMPLE

Solve $6x - 7 = 3x + 2$.*Solution.* $6x - 7 = 3x + 2$.Subtracting $3x$ from each member,

$$3x - 7 = 2. \quad \text{Ax. II}$$

Adding 7 to each member,

$$3x = 9. \quad \text{Ax. I}$$

Dividing each member by 3,

$$x = 3. \quad \text{Ax. IV}$$

Checking the solution of an equation is often called *testing or verifying the result*.

Check. $6x - 7 = 3x + 2$.Substituting 3 for x ,

$$6 \times 3 - 7 = 3 \times 3 + 2.$$

Simplifying, $18 - 7 = 9 + 2$, or $11 = 11$.Hence x satisfies the original equation.

EXERCISES

Solve the following equations and verify the results:

- | | |
|------------------------|---------------------------------|
| 1. $x + 4 = 7$. | 11. $4m - 7 = 2m + 3$. |
| 2. $y - 8 = 5$. | 12. $3l - 5 = 7 - l$. |
| 3. $3x - 4 = 14$. | 13. $6 - b = 9 - 4b$. |
| 4. $5x - 2 = 3x + 6$. | 14. $2x = 6 - x$. |
| 5. $5t = 3t + 8$. | 15. $5 + 3t = t + 3$. |
| 6. $4t = t + 3$. | 16. $5t - 1 = 2t + 8$. |
| 7. $3x + 2 = x + 6$. | 17. $5x - 1 = 2x + 5$. |
| 8. $4t + 2 = 3t - 4$. | 18. $n + 14 = 2 - 3n$. |
| 9. $4t + 2 = 2t$. | 19. $4w - 2 = 7 + w$. |
| 10. $4t + 2 = t + 5$. | 20. $4t + 3 - t + 5 = t - 10$. |

21. $3 + 2z - 1 = 12 - 3z$

22. $4m - 3 + 3m - 4 = 6m - 8$

23. $36 + 8m + 4 - 9m = 2m + 10$

24. $3m + 12m + 17 - m + 4m = 107$

25. $4h - 3 = 5h - 16 - 43h - 71$

26. $6x - 3 - 7x - 2 = 4x + 7 - 3x - 4$

27. $4y + 6 - 3y + 8 = 2y + 7 - 3y - 8 + 5y$

28. $3x + 7 - 4x + 2 + 5x = 2x + 4 - 5x + 7 - 3x + 22 + 6x$

ORAL EXERCISES

1. The side of a rectangle is x feet. How long is the side of a rectangle 3 feet shorter? of one 4 feet shorter? of one 5 feet shorter?

2. One rectangle is n feet long. How long is a rectangle 3 feet longer? 4 feet longer? 5 feet longer?

3. The side of a square is x feet. What is the side of a square 2 feet shorter?

4. A rectangle contains x square feet. How many square feet does it contain after it is diminished by 30 square feet? by 50 square feet? by 100 square feet?

5. A park contained x acres. How large was it when it was increased by 200 acres?

6. The area of a triangle is m square feet. What is the area of a triangle twice as large? 3 times as large? 4 times as large?

7. A triangle contains 10 square feet. What is the area of a triangle x times as large?

8. A bin contained t tons of coal. Four tons were removed. How much coal remained?

9. A bin contained 15 tons of coal after x tons were removed. How many tons did it contain at first?

10. The area of a circular pond is k square yards. What is the area of a pond that contains 200 square yards more? 250 square yards more? x square yards more?

11. The area of a circular pond containing x square feet is increased by 30 square feet. How many square feet does it now contain?

12. At the end of a dry summer the area of a certain lake whose area was originally x acres was found to have diminished 300 acres. What was its new area?

13. The volume of a pyramid is x cubic yards. What is the volume of a pyramid 3 times as large? 5 times as large?

Express the following statements as equations:

14. The sum of 3 and x is 10.

15. The sum of x and 2 is 12.

16. x is equal to 5 increased by 2.

17. x diminished by 5 equals 7.

18. x is equal to 10 diminished by 6.

19. x increased by 5 equals 3.

20. Three times x increased by 4 equals 16.

21. Three times a is 12 more than 15. } $a + 11 + 12$

~~22.~~ Four times x diminished by 8 equals two times x .

23. One half of a diminished by 5 equals 7.

24. One third of a increased by 10 equals four times a decreased by 1.

25. $2x$ increased by 5 is the same as $3x$ divided by 4.

26. Four times x diminished by 7 equals three times x added to 3.

27. One half of x increased by 5 equals two times x diminished by 10.

EXAMPLE

In his will a man bequeaths \$7000 to his wife and daughter with the provision that the wife is to receive \$1000 more than twice the amount received by the daughter. How much does each receive from his estate?

Solution. By the conditions of the problem the wife and daughter together receive \$7000 and the wife receives \$1000 more than twice that received by the daughter. The problem requires that the amount received by each be determined.

(In order fully to grasp the conditions of a problem, it is frequently necessary to make such a restatement as the above, although usually it need not be written down.)

The implied equation is:

Amount received by the wife + amount received by the daughter = \$7000.

Let d = number of dollars received by the daughter.

Then $2d + 1000$ = number of dollars received by the wife.

Hence $2d + 1000 + d = 7000$,

$$3d = 6000,$$

$d = 2000$, the number of dollars received by the daughter.

$2 \times 2000 + 1000 = 5000$, the number of dollars received by the wife.

Check. $\$5000 + \$2000 = \$7000$.

$$\$5000 = 2 \times \$2000 + \$1000.$$

The preceding solution leads to the following

RULE. *In the solution of problems involving the use of simple equations the following steps are necessary:*

1. *Read the problem carefully and find the statement which will later be expressed in an equation.*
2. *State briefly the implied equation in words, using the symbols of operation $+$, $-$, $\times$, $\div$, and the equality sign.*
3. *Represent the unknown numbers by means of numerals and letters.*
4. *Express the conditions stated in the problem as an equation involving these symbols.*
5. *Solve the equation.*
6. *Check, by substituting in the problem the values found for the unknowns.*

PROBLEMS

1. One boy has 10 more marbles than another. Together they have 52. How many has each?
2. Two boys caught 27 fish. One caught 5 more than the other. How many did each catch?
3. Two girls sell 37 bags of pop corn at a bazaar. One sells 3 more than the other. How many does each sell?
4. The perimeter of a rectangle is 46 feet. It is 3 feet longer than it is wide. What are its dimensions?
5. The perimeter of a rectangle is 50 feet. It is 4 times as long as it is wide. What are its dimensions?
6. One number is 4 times another number. Their difference is 36. Find the numbers.
7. A rectangle is 10 feet longer than it is wide. Its perimeter is 70 feet. What are its dimensions? *14 x 20*

8. One circle is half as large as another. Their combined areas are 30 square feet. What is the area of each?

9. The perimeter of a triangle is 41 yards. The first side is 4 yards shorter than the second, and the third is 3 yards less than the second. What is the length of each side?

10. A flagpole is half as tall as the building on which it stands. The flag at full mast is 150 feet above the ground. How long is the pole?

11. The sum of three numbers is 77. The second is twice the first and the third 5 more than three times the first. What are the numbers?

12. One fish pool is twice as large as another. In these two pools 72 fish are to be placed in numbers proportional to the size of the pools. How many fish should be placed in each pool?

13. A poultryman has three yards. The first is three times as large as the second, and the second is twice as large as the third. He wishes to separate his flock of 360 chickens into flocks proportionate to the size of the yards. How many chickens should he place in each yard?

14. A farmer's flock of 300 sheep is to be placed in three fields so that the second shall contain twice as many as the first and the third three times as many as the first. How many will there be in each field?

15. A flock of 100 sheep is to be divided into two flocks so that one flock shall contain 12 more than the other. How many sheep will there be in each flock?

16. Twenty-seven crates can be loaded on two trucks. One truck holds twice as many as the other. How many crates will each hold?

side 1 side
2-4 = 4
4-3 3-x

3-4-7-4720



John Wallis

17. Five hundred crates are to be packed by the workmen in two warehouses. Warehouse No. 1 has three times as many workmen as Warehouse No. 2. How many crates should be sent to each warehouse so that the work will be evenly divided among the men?

18. Three groups of men are employed to clean the streets after a snowstorm. The first group is twice as large as the second, and the third is as large as the first and second combined. There are 30 miles of street. How many blocks should each group clean if there are 10 blocks to a mile?

HINT. Let x denote the number of blocks that the second group can clean. Then $2x$ is the number of blocks that the first group can clean.

BIOGRAPHICAL NOTE

JOHN WALLIS. Among those who introduced and helped to standardize the modern algebraic symbolism was John Wallis, an Englishman. He was the son of a clergyman, and also took holy orders himself. Like most scholars of his day, he did not confine his interest to any one subject. He was at various times an instructor in Latin, Greek, and Hebrew, and for many years was professor of mathematics at Oxford. He also invented a method of teaching deaf mutes to talk.

During the wars between Charles I and Cromwell, Wallis's sympathies were with Cromwell, and he was of great service in reading royalist dispatches written in cipher. In fact, he was one of the most famous cryptologists of his day.

Wallis did not become interested in mathematics till the age of thirty-one, but devoted himself to the subject for the rest of his life. One of the earliest and most important books on algebra ever written in English was his treatise published in 1685. It contains a brief historical sketch of the subject, which is unfortunately not entirely accurate, but his treatment of the theory and practice of arithmetic and algebra has made the book a standard work for reference ever since.

REVIEW EXERCISES

1. Add $x^2 + xy - y^2$, $z^2 - yz - y^2$, $xz + z^2 - x^2$.
2. Add $x^2 + y^2 - 2xy$, $2z^2 - 4y^2 - 3yz$, $2x^2 - 4z^2 - 3xz$.
3. Add $3a^2 - 10b^2 + 5c^2 - 8bc$, $-a^2 + 4b^2 - 10c^2 - 3ab$, $c^2 + 11bc + 7ac - 5ab$, $4c^2 - 5bc + ac$, $-2a^2 + 6b^2 + 9ac + bc$.
4. Add $3x^2 + y^2 - 3yz + z^2$, $4xy + 3y^2 - 3yz$, $-2x^2 + 2xy - y^2 - 3z^2$.
5. Given $a = -5x + 3y + 2z$, $b = -5y + 3x + 2z$, and $c = -4z + 2x + 2y$, show that $a + b + c = 0$.
6. Given $x = b - 2c + 3a$, $y = -c - 2a - 3b$, and $z = -a + 2b + 3c$, show that $x + y + z = 0$.
7. From the sum of $4x^3 + 3x - 7$, $2x^2 - 3x - 3x^3 + 1$, and $-5x^3 - 2x + x^2 + 9$, subtract the sum of $2x^3 - 11x$ and $7x^3 - 5x^2 + 3 + 2x$.
8. The perimeter of a rectangle is 84 yards. The length is 6 yards more than twice the width. Find the length and the width.
9. The length of a rectangle is 4 feet less than five times the width. The perimeter is 88 feet. Find the dimensions.
10. A man is 8 years older than his wife; the age of their son is one fourth that of his mother. Their combined ages are 62 years. How old is each?
11. Give examples illustrating: (a) factors; (b) exponents; (c) coefficients.
12. Write five monomials; five binomials; five trinomials.
13. Simplify $4 + 7 - 3 + 4 \div 2 - 9 \div 3 + 6 \times 4 - 8 + 3$.
14. Simplify $10 + 3 \times 2 - 8 + 4 + 10 \div 5 + 18 + 6 \times 2 - 30 \div 2$.
15. What is the area of a rectangle if the length is twice the width, and the perimeter is 36 feet?

16. It is decided to build a circular track one-half mile long. What must be the radius of the circle? ($C = 2\pi r$, where $\pi = \frac{22}{7}$ and r is the radius.)

17. There are two numbers such that the first is five times the second, and the first is 32 more than the second. Find the numbers.

18. A father who is four times as old as his son is also 24 years older than his son. Find the age of each.

19. There are two numbers such that the larger is twice the smaller. If 12 is added to the larger, it will be six times as large as the smaller. Find the numbers.

20. A circular pond is being constructed in a certain park. Assuming that the bottom is level, how much water will it hold if the pond is 70 feet in diameter and is filled with water to the depth of 5 feet? What would it cost to fill it with water at ninety cents per 500 cubic feet? ($V = \pi r^2 h$.)

21. Find the horse power (H) of a 6-cylinder engine in which the diameter (d) of a cylinder is 5 inches, if the formula for the power of the engine is $H = d^2 n / 6$, where n is the number of cylinders.

22. Some men standing on the edge of a perpendicular cliff wished to know how far it was to the stream of water at the foot. They dropped a rock over the cliff and saw the splash 4 seconds later. How high was the cliff? ($S = \frac{at^2}{2}$, where $a = 32$, and t = the time in seconds.)

23. A certain rectangular playground is twice as long as it is wide. Its perimeter is 270 yards. Find the length and width.

CHAPTER VI

SUBTRACTION

37. Subtraction of monomials. In arithmetic it was found that subtracting 7 from 10 involved finding what number added to 7 would give 10. Similarly, to subtract $-2a$ from $+7a$, we find the number $+9a$, which when added to $-2a$ will give $+7a$.

EXAMPLES

1. From	$+ 10 a$	3. From	$- 10 ay^4$
take	$\underline{+ 5 a}$	take	$\underline{+ 3 ay^4}$
Result	$+ 5 a$	Result	$- 13 ay^4$
2. From	$7 ax$	4. From	$- 12 x^2y^4$
take	$\underline{- 3 ax}$	take	$\underline{- 3 x^2y^4}$
Result	$10 ax$	Result	$- 9 x^2y^4$

These examples illustrate the

RULE. To subtract one number from another, change the sign of the subtrahend and add.

EXERCISES

Subtract the lower number from the upper :

- | | | |
|--|--|--|
| <p>1. $\begin{array}{r} + 10 a \\ - 3 a \\ \hline \end{array}$</p> | <p>3. $\begin{array}{r} 10 a^2xy^2z^3 \\ - 10 a^2xy^2z^3 \\ \hline \end{array}$</p> | <p>5. $\begin{array}{r} 10 a^3 \\ - 3 a^3 \\ \hline \end{array}$</p> |
| <p>2. $\begin{array}{r} 4 a^2xy^2 \\ - 5 a^2xy^2 \\ \hline \end{array}$</p> | <p>4. $\begin{array}{r} 7 ay \\ + 2 ay \\ \hline \end{array}$</p> | <p>6. $\begin{array}{r} 3 x^2y^4 \\ - 5 x^2y^4 \\ \hline \end{array}$</p> |

$$\begin{array}{r} 7. \quad 6ax^2 \\ - 3ax^2 \\ \hline \end{array}$$

$$\begin{array}{r} 9. \quad -6ax^2 \\ - 3ax^2 \\ \hline \end{array}$$

$$\begin{array}{r} 11. \quad 6a^3y^4 \\ - 8a^3y^4 \\ \hline \end{array}$$

$$\begin{array}{r} 8. \quad 4axy^4 \\ - axy^4 \\ \hline \end{array}$$

$$\begin{array}{r} 10. \quad 17xyz \\ - 12xyz \\ \hline \end{array}$$

$$\begin{array}{r} 12. \quad -7ay^3 \\ - 5ay^3 \\ \hline \end{array}$$

Subtract:

$$13. -3ax \text{ from } -7ax.$$

$$17. 9ab^3 \text{ from } -3ab^3.$$

$$14. -6ax \text{ from } -2ax.$$

$$18. -6a^2b^4 \text{ from } 3a^2b^4.$$

$$15. 4ay^3 \text{ from } -3ay^3.$$

$$19. 10axy^3 \text{ from } 4axy^3.$$

$$16. -7ax^2 \text{ from } 3ax^2.$$

$$20. 3(x+y) \text{ from } 6(x+y).$$

$$21. 7(a-b) \text{ from } -4(a-b).$$

$$22. -3(w+v) \text{ from } -2(w+v).$$

$$23. -11(x^2-y^2) \text{ from } 5(x^2-y^2).$$

$$24. 2\sqrt{x+3} \text{ from } 3\sqrt{x+3}.$$

$$25. 5\sqrt{x^2-y^2} \text{ from } -2\sqrt{x^2-y^2}.$$

38. Subtraction of polynomials. Subtraction of one polynomial from another is illustrated in the following

EXAMPLE

Subtract $3x - 4y - 5z + 4$ from $x + 7y - 4z$.

Solution. $x + 7y - 4z$ = the number from which something is subtracted, or the minuend.

$3x - 4y - 5z + 4$ = the number subtracted, or the subtrahend.

$$\begin{array}{r} -2x + 11y + z - 4 = \text{the difference.} \end{array}$$

Check. Difference + subtrahend = the minuend. Adding the difference and the subtrahend in the solution above, we obtain $x + 7y - 4z$, which is the minuend.

This example illustrates the

RULE. Write the subtrahend under the minuend, so that similar terms are in the same column.

Then, changing mentally the sign of each term in the number to be subtracted, add the similar terms algebraically.

CHECK. As in arithmetic, difference + subtrahend = minuend.

EXERCISES

Subtract the first number from the second :

1. $a + 3$, $a + 4$.

7. $6xy$, $4xy - b$.

2. $a - 4$, $a - 7$.

8. $7a^2x - 4y$, $3a^2x$.

3. $10x + 7$, $7x - 10$.

9. $6a^3$, $5a - 7a^3$.

4. $3a$, $3a - 2$.

10. $5a - 7a^3$, $6a^3$.

5. $3a - 2$, $3a$.

11. $4ab - 7$, 0 .

6. $4xy - b$, $6xy$.

12. 0 , $4ab - 7$.

Subtract the first polynomial from the second and check the work :

13. $x + 2y - 3z$, $3x + 2y + 2z$.

14. $3x - 7y + 3z$, $3x - 4y - 5z$.

15. $2a - 3b$, $5a + b + 3c$.

16. $3b - 4c + 5d$, $3b - 5c$.

17. $2x^2 - 2x - 5$, $3x^2 + 7x - 3$.

18. $a + x - y$, $b + x + y$.

19. $4a - 3b + 2c + 7$, $5a - 2b + 6c$.

20. $2a - 2c - 4$, $a - 3b + 2$.

21. $3a - 4x + 5c$, $5x - 4a - 10$.

22. $x + y - z$, $z - x - y$.

23. $3a^2 - 4ab + 2b^2$, $4b^2 - 6a^2 + 5ab$.

24. $3a^3 - 3a^2b - 3ab^2, 4a^2b - 5ab^2 - b^3.$

25. $c^3 - d^3 - 3c^2d + 3cd^2, 5c^2d + 4d^3 + 3c^3 - 2cd^2.$

26. $3c^3 + 6cd^2 - 2a^2b - 6c^2d, 3ab^2 - 3a^2b - 6c^2d + 6cd^2.$

27. $x^5 + 1 - 2x + 3x^4 + x^3 + 3x^2, 6x^5 + 8x - 5x^2 + 16 + 6x^3 - 4x^4.$

28. $6x^5 + 2 - x^2 - 9x^4, 4x - 3x^2 - 14 + 8x^5.$

29. $3x^2 + 9x^3 + 3 - x, 7 + 5x^5 - x + x^3 - 2x^2 - 4x^4.$

30. Subtract the sum of $b^2 + 2bc - 1$ and $b^2 + 12bc - 20$ from $b^2 + 13bc - 30.$

31. Subtract the sum of $b - 3c + d$ and $4b + 5c - 6d + 4$ from $b - c + d - x.$

32. From the sum of $7x + 4x^2y - 16xy^2$ and $-8x - 13xy^2 + 8yx^2$ take $12x - 4x^2y + 8y^2x.$

33. From the sum of $5abc^2 - 4ab^2c + 3a^2bc$ and $5abc^2 - 4ab^2c - 3a^2b^2c$ subtract $2abc^2 - 4a^2bc + 8ab^2c.$

34. From the sum of $4x - 5xy - 3z$ and $8xy - 5z - 4x$ take the sum of $4z - 3xy - 2a^2bc$ and $9x - 5a^2bc - z.$

35. From the sum of $a^4 + a^2c^2 + c^4$ and $2c^4 - 3a^2c^2 - 4a^4$ take the sum of $2a^4 + 3c^4 + 3a^2c^2$ and $a^4 - 2c^4 - 4a^2c^2.$

36. From the sum of $3a + 5b - c$ and $3a - 5b + c$ take the sum of $2a - b + c$ and $b - c - 2a.$

37. From the sum of $7a^2 - 5a + 2$ and $5a - 7a^2 - 2$ take the sum of $3a^2 + 4a + 6$ and $-10 - 3a - 4a^2.$

38. From the sum of $3a + 4b - 3c, 2a - 3b + c, -c - 2b - 3a,$ and $4a - 3b - 4c,$ take the sum of $a - b + c, b + c - 3a,$ and $-c + 2a - 2b.$

39. From the sum of $6a^2y - 6ay^2 + y^3, a^3 - 3a^2y + 3ay^2,$ and $3ay^2 - 3a^2y,$ take the sum of $4a^2y + 3ay^2 + 2y^3, a^3 - 3ay^2,$ and $-2a^2y - y^3.$

CHAPTER VII

IDENTITIES AND EQUATIONS OF CONDITION

39. Kinds of equations. By definition, an equation is the statement of equality between two equal numbers or number symbols (p. 12).

⌚ A literal equation in which the two members are alike, term for term, or can be made so, is called an *identity*.

An identity is true for any value of the letter or letters. An equation involving only numbers is always an identity.

Thus $2 + 4 = 3 \cdot 2$, $4x + 2 = 3x + 2 + x$, and $2x + x = 4x - x$ are identities. The latter equation is an identity, for, by combining terms, the equation reduces to $3x = 3x$.

An equation is called an *equation of condition* if it is true only for certain values of the unknown involved.

The equation $4x = x + 9$ is true when $x = 3$, for if 3 is substituted for x , the equation becomes $4 \cdot 3 = 3 + 9$, or $12 = 12$. Clearly the statement is false if 2, 5, or any value other than 3 is put for x . The equation $4x = x + 9$ is true on condition that x be 3, and on no other.

ORAL EXERCISES

State which of the following expressions are identities:

1. $5 + 2 = 8 - 1$.

5. $2y = 4y - 3y + y$.

2. $2x = 6$.

6. $a + 7 = 8 + a - 1$.

3. $3b - 5 = 2b$.

7. $3w - 1 + w + 1 - 4w = 0$.

4. $x - 2 = 2x - 1$.

8. $w + 4 - 3w + 2 = 6w - 8$.

In the following, select from the group at the right of each equation those numbers which satisfy that equation :

- | | |
|-----------------------------------|-----------|
| 9. $4a - 12 = 0$. | 1, 2, 3. |
| 10. $2b = b + 2$. | 1, 2, 4. |
| 11. $d + 3 = 4d - 6$. | 0, -1, 3. |
| 12. $3a - 4 = 2a + 1$. | -2, 1, 5. |
| 13. $x + 6 = 3x - 10$. | 8, 5, -4. |
| 14. $m + 1 - 2m + 3 = 0$. | 7, 8, -3. |
| 15. $s^2 - 4s + 4 = 0$. | 2, 0, -2. |
| 16. $2a^2 - 8a + 8 = 0$. | 2, -2. |
| 17. $x^3 - 9x^2 + 27x - 27 = 0$. | 3, -3. |
| 18. $a^2 - 5a + 6 = 0$. | 2, 3, 0. |

40. Root of an equation. A number or a literal expression is called a *root of an equation* if, after substituting it for the unknown letter in the equation, the result is or can be reduced to an identity.

Thus 3 satisfies the equation $4x = x + 9$ and, similarly, $2a$ satisfies the equation $x - 5 = 2a - 5$.

✓ *A number or number symbol is called a root of an equation if it satisfies the equation.*

ORAL EXERCISES

Is the number at the right of each of the following equations a root of that equation?

- | | | | |
|-------------------------|----|--------------------------|-----|
| 1. $3a - 24 = 0$. | 8. | 5. $d^2 - 16 = 4d + 5$. | 7. |
| 2. $4x - 9 = x$. | 3. | 6. $m^2 + 7 = 6m - 1$. | 4. |
| 3. $5r - 12 = r$. | 2. | 7. $a^2 + 5a + 4 = 0$. | -2. |
| 4. $a^2 + 5a - 6 = 0$. | 4. | 8. $m + 3 = 2m - 1$. | -3. |

41. Transposition. In solving the equation $5x - 2 = 18$ we add 2 to each member. If we indicate this addition, the equation becomes

$$5x - 2 + 2 = 18 + 2.$$

In the first member, $-2 + 2 = 0$. Hence these two numbers may be omitted, and the equation becomes

$$5x = 18 + 2.$$

Comparing this with the original equation,

$$5x - 2 = 18,$$

we see that -2 has vanished from the first member of the original equation and $+2$ has appeared in the second member of the new equation. The number $+2$ has really been added to both terms of the equation.

Again, if we subtract $2y$ from both members of

$$6y = 2y + 12,$$

we get the equation $6y - 2y = 12$,

which differs from the original equation only in having $-2y$ in the first member instead of $+2y$ in the second. Hence a term may be omitted from one member of an equation if the same term with its sign changed from $+$ to $-$ or from $-$ to $+$ is written in the other member.

This process is called *transposition*.

Hereafter, instead of going through the details of subtracting a number from both members of an equation (or adding a number to both members), the student should use transposition. He should remember, however, that the transposition of a term is really the subtraction of that term from (or addition to) each member of an equation.

Like terms in the same member of an equation should be combined before transposing any term.

NOTE. Our word *algebra* is derived from the Arabic word for *transposition*. The process by which one passes from the equation $px - q = x^2$ to the equation $px = x^2 + q$ was known as *al-jabr*. This is the first word in the title of an Arabic book on algebra which was translated into Latin. For some reason only this part of the title remained, and by the early part of the seventeenth century *al-jabr*, or algebra, was the common name given to the whole subject.

ORAL EXERCISES

State the term or terms which must be added to both members of the following equations in order to transpose the underscored terms. What does each equation become after the operation is performed?

- | | |
|---|--|
| 1. $x - \underline{5} = 3.$ | 7. $3a + \underline{3} + 4a + \underline{5} = \underline{3}a + 4.$ |
| 2. $n + \underline{12} = 3.$ | 8. $5x - \underline{3} - 4x + \underline{7} = 8 - \underline{2}x.$ |
| 3. $x - \underline{4} = 2.$ | 9. $r^2 + 2r = -\underline{4}.$ |
| 4. $n^2 + 6n = \underline{4}n - 3.$ | 10. $3d - 2 = -\underline{2}d.$ |
| 5. $x^2 + 4x = \underline{5}.$ | 11. $5x - 8x = \underline{5}x - \underline{2}.$ |
| 6. $a^2 + 3a + 2 = \underline{2}a - \underline{3}.$ | 12. $6r - r^2 = \underline{2} - \underline{2}r^2.$ |

EXAMPLE

Solve the equation

$$32n - 13 + 7n + 9 - 4n = 7n + 16 + 12n + 12.$$

Solution. $32n - 13 + 7n + 9 - 4n = 7n + 16 + 12n + 12.$

Combining like terms, $35n - 4 = 19n + 28.$

Transposing, $35n - 19n = 28 + 4.$

Combining like terms, $16n = 32.$

Dividing by 16, $n = 2.$

Check. $32n - 13 + 7n + 9 - 4n = 7n + 16 + 12n + 12.$

Substituting 2 for n ,

$$64 - 13 + 14 + 9 - 8 = 14 + 16 + 24 + 12.$$

Combining,

$$66 = 66.$$

EXERCISES

Solve the following equations and check:

1. $4x - 2 = 3x + 2.$
2. $3x + 5 = x - 5.$
3. $-7y - 2 = 1 - 10y.$
4. $6x + 3 - 3x = 21.$
5. $3y + 5 = 6 + 2y.$
6. $4h + 1 - 2h = 5.$
7. $16 + 4h - 4 = h + 12.$
8. $4h - 15 = 15 - h - 5.$
9. $5h + 11 - 2h = 6 - 4.$
10. $2h - 1 = 29 + 7h.$
11. $3h + 9 + 5h - 33 = 0.$
12. $3x - 20 + 8x - 24 = 0.$
13. $6h - 19 + 2h + 3 = 0.$
14. $h + 2 - 5h = 9h + 15.$
15. $4h + 3 = 15 + 2h + 6.$
16. $3h + 5 + h + 3 = 0.$
17. $9 - 8n + 2 = 3 - 4n.$
18. $3 - 5n + 2 = 5 + 7n.$
19. $8n - 3n = 15n + 4 - 13n.$
20. $8 + 7n - 13 = n - 27 - 5n.$
21. $3n - 15 - 10n - 9 + 16n - 21 = 0.$
22. $18 + 5n - 6 - 2n + 1 + 3n - 25 = 0.$
23. $0 = 18 - 4n + 27 + 9n - 3 + 16n.$
24. $5t - 8 + 4t + 5 = 7t - 3 - 2t + 5.$
25. $0 = 4p - 15 - 11p - 18 + 16p - 17.$
26. $5p - 6 + 3p + 18 - 2p - 25 + 1 = 0.$
27. $3p + 18 - 2p - 3 = 7p - 5 - 4p + 8.$
28. $0 = 6 - 8p + 8 - 2 + 4p.$
29. $3p + 5 + 8p + 50 = 0.$
30. $2 - 2x + 3x - 3 = 2x - 2.$
31. $2d + 3 + 3d + 5 = 4d + 6.$
32. $2a^2 + 3a + 4 = 2a^2 + 2a.$
33. $3s^2 + 12s = 3s^2 + 6s - 12.$
34. $a + 4 + a^2 + 2a - 6 = a - 3 + a^2.$
35. $n^2 - n + 1 = n^2 + n - 1.$
36. $r^4 - r + 6 = r^4 + r - 6.$

ORAL EXERCISES

Represent a number :

1. Greater by 5 than 4.
2. Greater by 5 than x .
3. Greater by 5 than y .
4. Greater by x than y .
5. Greater by x than twice y .
6. Nine more than y .
7. Nine more than $x + y$.
8. Two less than 4.
9. Four less than x .
10. Four less than $x + y$.
11. Less by 7 than x .
12. Less by y than x .
13. Less by y than twice x .
14. Less by y than three times x .
15. Three times p .
16. One half as great as p .
17. Three more than half of p .
18. p less than three times x .
19. p less than $x + y$.
20. p more than y less z .
21. One part of 10 is 6. What is the other part?
22. One part of x is 6. What is the other part?
23. One part of x is 10. What is the other part?
24. One part of n is a . What is the other part?
25. One part of a is n . What is the other part?
26. One part of $n + p$ is 1. What is the other part?
27. The sum of two numbers is 15. One is 10. What is the other?
28. The sum of two numbers is x . One of them is 3. What is the other?
29. The difference between two numbers is 5. The less number is 3. What is the other?

30. The difference between two numbers is 12. The less number is 17. What is the other?

31. The sum of two numbers is n . One number is p . What is the other?

32. The sum of two numbers is n . One of the numbers is three times as large as p . What is the other?

33. The difference between two numbers is 12. The greater number is 17. What is the less?

34. The difference between two numbers is x . The greater number is y . What is the less?

35. The difference between two numbers is 17. The less number is 4. What is the greater?

36. The difference between two numbers is x . The less number is p . What is the greater?

37. By how much does 12 exceed 8? 15 exceed 10? 17 exceed 11? x exceed 3? x exceed 10? x exceed y ?

38. How much less is 6 than 10? x than 20? x than y ?

39. By how much does $n + 6$ exceed 6? $n + 6$ exceed p ? $3n + 6$ exceed $2n$? $5n - 3$ exceed $4n$?

42. Translation of problems into equations. In the solution of problems the writing of the equations is merely translating the statement of the problem from ordinary language into the language of algebra. Sometimes it is possible to translate the original statement of the problem word by word into algebraic symbols.

For example:

Three times a certain number, diminished by 6,
$3 \quad \times \quad n \quad - \quad 6$
gives the same result as the number increased by 14.
$= \quad n \quad + \quad 14$

PROBLEMS

Solve the following problems, making direct translations into the symbols of algebra whenever possible:

1. What number increased by 12 is equal to 30?
2. What number increased by 20 is equal to 30?
3. What number increased by 10 is equal to twice the number?
4. What number diminished by 20 is equal to 9?
5. What number diminished by 12 is equal to twice the number?
6. To what number must 20 be added so that the result may be 30?
7. To what number must 20 be added so that the result will be 15?
8. From what number must 10 be subtracted so that the result may be three times the number?
9. A certain number is doubled and the result increased by 10. The sum is 14. What is the number?
10. Three times a certain number less 25 equals twice the number less 15. What is the number?
11. Five times a certain number increased by 14 equals eight times the number diminished by 4. Find the number.
12. Four times a certain number increased by 7 equals five times the number diminished by 7.
13. What number is as much less than 80 as it is greater than 26?
14. What number is as much less than 100 as it is greater than 50?

15. What number exceeds 20 by twice as much as 50 exceeds the number?

16. A certain number added to 17 gives the same result as that obtained when twice the number is subtracted from 62. What is the number?

17. Twice a certain number added to 30 gives the same result as three times the number subtracted from 90. What is the number?

18. The sum of two numbers is 54, and their difference is 12. What are the numbers?

Solution. Two numbers are to be found. Their sum is 54, their difference 12.

Greater number + less number = 54.

If n = less number,
then $n + 12$ = greater number.

Placing these symbols in the principal statement above, we have

$$n + 12 + n = 54.$$

Combining, $2n + 12 = 54.$

Transposing, $2n = 54 - 12.$

Combining, $2n = 42.$

Dividing by 2, $n = 21,$

and $n + 12 = 33.$

Check. $21 + 33 = 54, 33 - 21 = 12.$

19. The sum of two numbers is 75, and their difference is 41. What are the numbers?

20. The sum of two numbers is 14, and their difference is 30. What are the numbers?

21. The sum of two numbers is 50, and their difference is three times the smaller number. Find the numbers.

22. The sum of two numbers is 10. The greater is 14 more than the less. What are the numbers?

23. The sum of two numbers is 57, and one exceeds the other by 15. What are the numbers?

24. The sum of three numbers is 115. The second is 7 greater than the first, and the third is 8 greater than three times the first. Find the numbers.

25. The sum of three numbers is 37. The first is 5 less than the second, and the third is 2 more than twice the second. What are the numbers?

26. A triangle whose perimeter is 45 inches has sides in the ratio of 2, 3, and 4. Find the sides.

27. The first side of a triangle is 3 inches less than the second, and the third is 2 inches more than twice the first. The perimeter is 37 inches. What are the sides?

28. A rectangle whose perimeter is 36 feet is twice as long as it is wide. What are its dimensions?

29. A rectangle whose perimeter is 56 feet is three times as long as it is wide. What are its dimensions?

30. A certain garden requires 320 yards of fencing for the four sides. The garden is four times as long as it is wide. Find the length and width of the garden.

31. A real estate dealer can afford to spend only \$80 per week for office help. If the weekly salary of the office boy is \$10 and that of the stenographer is three fourths as much as that of the bookkeeper, how much should he pay each?

32. A man can allow his three children all together \$2 a week for spending money. James needs 50 cents per week more than Ruth, and Charles requires only half as much as Ruth. What allowance will each child receive?

33. Mr. James paid \$12 for tickets for himself and his wife and half-fare tickets for John, aged 10, and Frank, aged 7. What is the price of one full-fare ticket?—

34. Find two consecutive numbers whose sum is 53.

35. Find three consecutive numbers whose sum is 66.

36. Find four consecutive numbers whose sum is 74.

37. Find two consecutive odd numbers whose sum is 92.

HINT. Let n = the first number, and $n + 2$ the second number.

38. Find two consecutive odd numbers whose sum is 108.

39. Find three consecutive odd numbers whose sum is 117. *use 2 + 4*

40. Find four consecutive odd numbers whose sum is 192.

41. Find two consecutive even numbers whose sum is 46.

42. Find three consecutive even numbers whose sum is 108. *2 - 4*

43. Find five consecutive even numbers whose sum is 60.

CHAPTER VIII

PARENTHESES

43. Removal of parentheses. In solving exercises and problems it is often necessary to treat several terms as though they were one term. This is done by inclosing the various terms in a parenthesis.

Thus, if n represents an even number, the sum of three consecutive even numbers might be represented as follows :

$$n + (n + 2) + (n + 4).$$

Sometimes it is necessary to inclose a number in a parenthesis along with other terms inside a second parenthesis. To avoid confusion in such cases, different parentheses, such as brackets [] and braces { }, are used. Parentheses, brackets, and braces are sometimes called symbols of aggregation. For convenience, where no confusion arises, braces and brackets are spoken of as parentheses.

In the solution of equations and in other algebraic work it is often necessary to remove all signs of grouping. This removal, although it depends upon the principles governing addition and subtraction, must be given some special study if the required speed and accuracy are to be attained.

44. Removal of parentheses preceded by the sign +. The value of $10 + (4 - 1)$ is the same as that of $10 + 4 - 1$. Similarly, $a + (b - d) = a + b - d$.

The plus signs preceding the parentheses $(4 - 1)$ and $(b - d)$ in the above expressions disappear with the parentheses, but the signs of the terms within the parentheses, whether expressed or understood, are always supplied when the parentheses are removed, as in the case of 4, 1, b , and d , in the above expressions. Since in adding one expression to another the signs of the terms added are not changed, we have the following

PRINCIPLE. A parenthesis and the plus sign before it may be removed from an expression without changing the signs of the terms which were inclosed by the parenthesis.

45. Removal of parentheses preceded by the sign $-$. In the expression $10 - (5 - 3)$ the sign before the binomial shows that $(5 - 3)$ is to be subtracted from 10. To subtract $(5 - 3)$ we change the signs of the terms subtracted and add the resulting terms to the minuend.

Thus, $10 - (5 - 3) = 10 - 5 + 3 = 8$. This is correct, since $10 - (5 - 3) = 10 - 2 = 8$.

Similarly, $a - (c - d)$ becomes $a - c + d$ when the signs of $(c - d)$ are changed and the result is added to a .

The minus signs preceding the parenthesis in $10 - (5 - 3)$ and $a - (c - d)$ disappear when the parentheses are removed. The plus signs understood before 5 and c in the parentheses are changed, as is the sign of each term within the parentheses, when we write $10 - (5 - 3) = 10 - 5 + 3$, and $a - (c - d) = a - c + d$. Since, in subtracting one expression from another, we add with their signs changed the several terms to be subtracted, we have the following

PRINCIPLE. A parenthesis and the minus sign before it may be removed from an expression, provided the sign of each term which was inclosed by the parenthesis is changed.

ORAL EXERCISES

Remove the parentheses and where possible combine terms in the resulting expressions:

1. $7 + (4 - 1).$

6. $x - (w + v).$

2. $7 - (4 + 1).$

7. $x - \{w - v\}.$

3. $7 - (-4 + 1).$

8. $x + (-w - v).$

4. $x + [w + v].$

9. $x - [-w - v].$

5. $x + \{w - v\}.$

10. $3x - (x - 2y + 5).$

46. Removal of two or more parentheses. If neither of two parentheses is within the other, both parentheses may be removed at the same time, proper regard being given to the principles governing the removal of each.

If one parenthesis incloses another, the inner parenthesis should be removed first, in accordance with the following

RULE. Rewrite the expression, omitting the innermost parenthesis, changing the signs of the terms which it inclosed if the sign preceding it is minus and leaving them unchanged if it is plus.

Combine like terms within the new innermost parenthesis.

Repeat these processes until all the parentheses are removed.

EXAMPLE

Remove the parentheses and collect terms:

$$12 - [15 - 8a + (18 - 9a)] + 2.$$

Solution. Removing the inner parenthesis first,

$$12 - [15 - 8a + 18 - 9a] + 2.$$

Collecting terms, $12 - [33 - 17a] + 2.$

Then removing the remaining parenthesis,

$$12 - 33 + 17a + 2.$$

Collecting terms, $17a - 19.$

EXERCISES

Remove the parentheses and combine like terms:

1. $10 - (6 - 3) - 4.$
2. $12 + (7 - 4) - (8 - 6).$
3. $(8 - 5 + 2) - (4 - 2) + 7.$
4. $10a - (6a - 3a) + (7a - a).$
5. $(2n - 5p) - (2n - p - 6n).$
6. $n - (p - q) + \{2q - 3p\}.$
7. $n - p - (m - q) + (n - p) - (m - n).$
8. $(n - p) - \{2p - 3n\} + (n - 4p).$
9. $n - (n - p - q) - (3q + p + 1) + (n - 5).$
10. $(n - p) - (3n - 2p) + [2p - n] - 3p.$
11. $8 - [9 - (4 - 10)] - (15 - 24).$
12. $9 - [5 - (4 - 6)] - (8 - 15).$
13. $14 - (8 - 16) - [(4 - 1) - 8].$
14. $15 - [3 - (2 - 5)] + (3 - 5) - (8 - 3).$
15. $p - [2n - (3p - 5)].$
16. $[(n + p) - n] - p.$
17. $[(a + p) + (a - 5p)] - 4a.$
18. $(-r - s) - [(2r - s) + (r + s)].$
19. $-(-a - p) - [(-a + 5p) + (-2p - a)].$
20. $a + [3a - (4a - 2b)] + (3b - 2a).$
21. $a + [4a - (3x - 2a)] - (4a - 5x).$
22. $2a - [6a - (5n - 4a)] + (8a - 7n).$
23. $(5x - 6y) - [-4x - (4z - y)] - 2z.$
24. $[3p - (2n + q)] - [- (3n - 2p) + 5p].$

In the following, remove the parentheses, retaining the brackets, and simplify results as much as possible:

25. $[(n + p) + r], [(n + p) - r]$. *[n + p + r] [n + p - r]*

26. $[4n + (3r - 5p)], [4n - (3r - 5p)]$.

27. $[(n - p) + (p - 2n)], [(n - p) - (p - 2n)]$.

28. $[(3n - 2p) + (2p - 3n)], [(3n - 2p) - (2p - 3n)]$.

29. $[(n - 2p) + (3r - s)], [(n - 2p) - (3r - s)]$.

30. $[(4n - 3) + (5p - 7)], [(4n - 3) - (5p - 7)]$.

31. $[(n^2 - p^2) + (r^2 - 2p^2)], [(n^2 - p^2) - (r^2 - 2p^2)]$.

47. Inclosing terms in parentheses. Obviously, $12 + 7 - 3 = 12 + (7 - 3)$, for each equals 16.

Similarly, $c + d - f = c + (d - f)$.

That the expressions in the illustration above are equal may be seen by removing the parentheses according to the first principle on page 84.

Thus, $12 + (7 - 3) = 12 + 7 - 3$, and $c + (d - f) = c + d - f$.

This process of inclosing terms within a parenthesis illustrates the

PRINCIPLE. *One or more terms may be inclosed in a parenthesis preceded by a plus sign without changing the sign of any of the terms.*

ORAL EXERCISES

Inclose in a parenthesis preceded by a plus sign the last three terms in each of the following:

1. $4 + (7 - 6 - 3)$

5. $a + b - c - d$.

2. $8 + (4 + 7 - 2)$

6. $3a + 4b - c - d$.

3. $9 - 4 - 7 + 3$.

7. $3n - 4p - q - 3n$.

4. $a - b + c + d$.

8. $n^2 - 2p + p^2 + 1$.

The expression $15 + 7 - 4 = 15 - (-7 + 4)$, for each equals 18.

Similarly, $x + y - z = x - (-y + z)$,
and $x - y + z = x - (y - z)$.

That the right member in each of these cases is another form of the left may be seen by removing the parentheses according to the second Principle on page 84.

Thus, $15 - (-7 + 4) = 15 + 7 - 4$, the original expression, and $x - (y - z) = x - y + z$.

This process of inclosing terms in a parenthesis preceded by a minus sign illustrates the

PRINCIPLE. *One or more terms may be inclosed in a parenthesis preceded by a minus sign provided the sign of each term thus inclosed be changed.*

EXERCISES

Inclose in a parenthesis preceded by a minus sign the last three terms in the following:

1. $4 - 7 - 6 - 3$.

8. $n^2 + 2p - p^2 - 1$.

2. $7 + 3 + 6 - 2$.

9. $p^2 - 4n - n^2 - 4$.

3. $8 - 4 - 7 + 2$.

10. $p^2 - 2np + n^2 - n^2 + n + 1$.

4. $a - b + c + d$.

11. $p + q - r - 3n + 4m$.

5. $a - b - c - d$.

12. $p + q + r + n + m$.

6. $3p + 4n - q - r$.

13. $2p - q + 3r + 2p - q - r + qr$.

7. $3p - 4n + q - 3r$.

14. $4p^2 - 4q^2 + n^2 + 2pn + p^2$.

In the following, inclose in a parenthesis preceded by a plus sign all the terms containing x or y , and inclose in a parenthesis preceded by a minus sign all the other terms:

15. $x^2 - a^2 - 3a$.

17. $y^2 - b^2 + 2ab - a^2$.

16. $10a + x^2 - 25 - a^2$.

18. $20ab + x^2 - 4a^2 - 25b^2$.

$$19. x^2 - b^2 - 4b^2 + 2y^2 - a^2 - 2xy - y^2. \quad 2-2-4-1$$

$$20. -4ab + x^2 - b^2 + 4y^2 - 4a^2 - 4xy.$$

$$21. 4x^2 - a^2 - 8xy - 4 + 4a + 4y^2.$$

$$22. x^2 - 9b^2 - 10xy + 36ab - 36a^2 + 25y^2.$$

$$23. 4ab - 4a^2 + x^2 + 4xy - b^2 + 4y^2.$$

$$24. x^2 - 16xy - p^2 + 16p + 6xy^2 - 6x.$$

$$25. x^2 + a^2 + 2ab + 8xy + 2bc + 16y^2 - 2ac - b^2 - c^2.$$

REVIEW EXERCISES

In the first four exercises, is the number to the right a root of the corresponding equation?

$$1. x^2 - 7x + 6 = 2x + 4 - (8 + 2x). \quad 2.$$

$$2. a^2 - (2a - 3) = a^2 - (7 - 2a) + 6. \quad 3.$$

$$3. (4 - 2x) - (3x - 12) + (4x - 10) = 4x - 20 - (1 - 4x). \quad 3.$$

$$4. (3x + 6) - (4 - 7x) = 19x - (11x - 4). \quad 7.$$

5. Indicate the sum of $2x$ and $(x + 3)$; 3 and $(x - 5)$; $(7 + x)$ and $6x - 2$.

6. Indicate the difference found by subtracting $(4x - 7)$ from $(6x + 3)$.

7. Indicate $(9x + 2)$ diminished by $(3x - 1)$ plus $(2 - 5x)$.

8. Indicate the sum of $(3x + 2)$, $(5x + 6)$, and $(9x - 3)$ diminished by the sum of $(4x - 7)$ and $(3x + 2)$.

9. Indicate the ages of three boys if the first is twice as old as the second and the third 3 years younger than the second. Write the equation if the sum of their combined ages is 21.

10. Find three numbers whose sum is 34, if the first two are consecutive even numbers and the third is twice the first.

4-5

CHAPTER IX

MULTIPLICATION

48. Product of terms containing unlike letters. The student is familiar with the fact that the factors of a product may be taken in any order.

Thus, $3 \cdot 2 \cdot 4 = 4 \cdot 3 \cdot 2 = 2 \cdot 3 \cdot 4 = 24.$

Similarly, $a \cdot b = b \cdot a.$

Again, $2 m^2 \cdot 3 n^2 = 3 m^2 \cdot 2 n^2 = 2 \cdot 3 m^2 n^2 = 6 m^2 n^2.$

This illustrates the

RULE. To multiply terms containing numbers and letters, write the product of the numeric coefficients followed by the product of all the literal factors.

49. Product of terms containing like letters. By the definition of an exponent (p. 17), $x^2 = x \cdot x$, and $x^3 = x \cdot x \cdot x$.

Therefore $x^2 \cdot x^3 = (x \cdot x) \cdot (x \cdot x \cdot x) = x^5 = x^{2+3}.$

Similarly,

$a \cdot a^3 \cdot a^5 = a \cdot (a \cdot a \cdot a) \cdot (a \cdot a \cdot a \cdot a \cdot a) = a^9 = a^{1+3+5}.$

In like manner,

$$\begin{aligned} 3^2 \cdot 3^4 \cdot 3^5 &= (3 \cdot 3) \cdot (3 \cdot 3 \cdot 3 \cdot 3) \cdot (3 \cdot 3 \cdot 3 \cdot 3 \cdot 3) \\ &= 3^{11} = 3^{2+4+5}. \end{aligned}$$

Also, $np^2 \cdot p^3 = np^5 = np^{2+3},$

$$3 ab \cdot 2 a^2 = 6 a^3 b = 6 a^{1+2} b,$$

and $3 x^2 y z^2 \cdot 5 x y^3 = 15 x^3 y^4 z^2 = 15 x^{2+1} y^{1+3} z^2.$

Therefore we have the

PRINCIPLE. *The exponent of any letter in a product is equal to the sum of the exponents of that letter in the factors. This is expressed in general terms thus :*

$$n^a \times n^b = n^{a+b}.$$

The law of signs for the multiplication of positive and negative numbers, given on page 41, applies to literal terms as well.

Thus,

$$\begin{aligned} (+3n^2) \cdot (+5n^3) &= +15n^5, \\ (+3n^2) \cdot (-5n^3) &= -15n^5, \\ (-3n^2) \cdot (+5n^3) &= -15n^5, \\ (-3n^2) \cdot (-5n^3) &= +15n^5. \end{aligned}$$

For the multiplication of two monomials, we have the

RULE. *Obeying the rule of signs for the multiplication, write the product of the numeric coefficients followed by all the letters that occur in the factors, each letter having as its exponent the sum of the exponents of that letter in the factors.*

ORAL EXERCISES

Perform the following indicated multiplications :

- | | | |
|----------------------|-----------------------|---------------------------|
| 1. $(3)(-5).$ | 6. $n^2 \cdot n^3.$ | 11. $n^5 \cdot n^7.$ |
| 2. $(-2)(6).$ | 7. $n^2 \cdot n^5.$ | 12. $-n^2 \cdot (n^4).$ |
| 3. $(-7)(-3).$ | 8. $n^4 \cdot n^5.$ | 13. $-n^3 \cdot (-n^2).$ |
| 4. $(-4x)(3).$ | 9. $n^2 \cdot n^2.$ | 14. $-n^2 \cdot (+3n^3).$ |
| 5. $3x \cdot 2x.$ | 10. $n^3 \cdot n^3.$ | 15. $-n(2n^5).$ |
| 16. $(-2p^2)(3p^3).$ | 19. $(-3p^3)(-4p^4).$ | |
| 17. $(3p^4)(-2p^5).$ | 20. $(-6b^2)(4b).$ | |
| 18. $(4p^2)(-5p^4).$ | 21. $(-3x)(-2x^3).$ | |

- | | |
|--------------------------------|---------------------------------|
| 22. $(+ 2 a^2)(- 4 a^2 x)$. | 41. $(- 2 n)(- 4 n^2)$. |
| 23. $(- 4 a^3 x)(+ 3 x^3)$. | 42. $(5 m^4)(9 m^3)$. |
| 24. $(- 5 a^2 x)(- 4 a x^2)$. | 43. $(- 5 p)^3$. |
| 25. $(+ 3 a^2 r^3)(3 a r)$. | 44. $(n^8)(- 16 n)$. |
| 26. $(- 3 n)^2$. | 45. $(- 4 n^5)(- 6 n^3)$. |
| 27. $(5)(- 7 a)$. | 46. $(+ 3 p)^2$ <i>+ 9 p^2</i> |
| 28. $(3 a)(- 7)$. | 47. $(+ 3 p)^3$. |
| 29. $(- 2 n)^2$. | 48. $(4 p)(5 n)(6 n p)$. |
| 30. $(- 7 a)(- 10)$. | 49. $(3 p)(- 2 n)^2$. |
| 31. $(- 3 n p)^2$. | 50. $(- 3 n^2 p)^2$. |
| 32. $(3 a)(- 7 a)$. | 51. $(3 h^2)(- k)$. |
| 33. $(5 n p q)^2$. | 52. $(5 n^2 p)(- 2 n^3)$. |
| 34. $(- 10 x)(4 x)$. | 53. $(- 5 x^3 y^2)^2$. |
| 35. $(5 x)(- 4 x)$. | 54. $(- 5 x^3 y^2)^3$. |
| 36. $(- n)^2$. | 55. $(- n^4 p)(- n^2 p^4)$. |
| 37. $(- n)^3$. | 56. $(2 n p^2)^3$. |
| 38. $(- 2 n)^2$. | 57. $(5 n^3)(- 4 n^2)(- 3 n)$. |
| 39. $(- 2 n)^3$. | 58. $(- 2 k^2)(3 k^3)^2$. |
| 40. $(2 a^2 x)^2$. | 59. $(- 2 a^2)^3(5 a^3)^2$. |

50. Multiplication of a polynomial by a monomial. The multiplication of a polynomial by a monomial may be illustrated as follows:

$$4(5 + 3) = 4 \times 5 + 4 \times 3 = 20 + 12 = 32,$$

and $3(6 + 4) = 3 \times 6 + 3 \times 4 = 18 + 12 = 30.$

Similarly, $b(x + y) = bx + by,$

and $2 a(x + y) = 2 ax + 2 ay.$

50
100

100
100

200
100

100
100
100



Sir William Rowan Hamilton

For the multiplication of a polynomial by a monomial, we have the

RULE. *Multiply each term of the polynomial by the monomial and write the resulting terms in succession with their proper signs.*

The actual work of multiplication may conveniently be arranged as in the following

EXAMPLE

Multiply $2n^2 - 3ny + 4y - 3p - 6$ by $2np$.

$$\begin{array}{r}
 \text{Solution. } + 2n^2 - 3ny + 4y - 3p - 6 \\
 + 2np \\
 \hline
 4n^3p - 6n^2py + 8npy - 6np^2 - 12np
 \end{array}$$

BIOGRAPHICAL NOTE

SIR WILLIAM ROWAN HAMILTON. It is strange that of all the topics treated in this book the last to be thoroughly understood by mathematicians are those appearing in the first chapters. But in all the sciences it is often most difficult to answer the questions that at first sight seem quite obvious. Any child can ask what electricity is, but the wisest scientist cannot tell. He can only explain what electricity does. It is easy to ask how the earth came to be revolving around the sun with the moon revolving around it, but even the deepest students of astronomy differ in their theories of how it came to be. And so in mathematics, long after many of the more complicated processes of algebra were completely understood, the simple laws of operation of numbers were only slightly understood. One of the men who did most to clarify the nature of these laws was Sir William Rowan Hamilton (1805-1865). He was born in Dublin, Ireland, where he lived most of his life. He was a precocious boy, and at the age of twelve was familiar with thirteen languages. He devised kinds of numbers that do not follow the same laws as those that we use in algebra, and so threw a flood of light on the nature and properties of these common numbers. Most of his works are very advanced in character and are difficult to read.

EXERCISES

Multiply :

1. $x + 2$ by 3.
2. $x - 3$ by x .
3. $x^2 + 4$ by $3x$.
4. $2x^2 + 5$ by $2x$.
5. $x^2 - 3x$ by x^2 .
6. $n^2 - 3n + 2$ by $3n$.
7. $p^3 - 5p^2 + 3p - 1$ by p^2 .
8. $4n^2 - 2n - 4$ by n^2 .
9. $-4p^2 + 6p - 5$ by $6p^2$.
10. $2p^2 - 3p - 3$ by $-4p^2$.
11. $p^3 - 3p^2 - 4$ by $-5p^4$.
12. $3x - 4x^2 - 3x^3$ by $-3x^3$.
13. $7xy^2 - x + y$ by $2xy$.
14. $x^2 - 2xy + 4y^2$ by $-2xy$.
15. $n^4 - n^2p^2 + p^4$ by $-n^2p^2$.
16. $-n^2p^2 - 2np + 7q^2$ by $-4npq$.
17. $8x^3 - 7x^2 + 11x - 5$ by $-3x^3$.
18. $-8a^2 - 10ax + 40x^2$ by $5ax^3$.

Perform the indicated multiplications :

19. $8(2x - 3)$.
20. $4x(x - y)$.
21. $-7(3x - 6)$.
22. $-8(-3a + 2b)$.
23. $-2x(3x - 6)$.
24. $5x^2(8x^3 - 3x)$.
25. $-3(n^2 - 2n - 6)$.
26. $5np(p^2 - 6p + 9)$.
27. $-2n(np - rp - 3sp^2)$.
28. $(n^3 - an + a^2)(-3a^2n)$.
29. $-6bc(bx^2 - cx + d)$.
30. $3x^2(-2x + 3x^2 - 2x^4)$.
31. What is the area of a square whose side is x inches?
 $2x$ inches? $3x$ inches? $4x$ inches?
32. What is the area of a rectangle whose length is $3x$
 and whose width is $2x + 3$?
33. What is the volume of a rectangular solid whose
 dimensions are $x + 1$, $2x$, and $5x$?
34. The edge of a certain cube is $3x$ inches. What is the
 total area of the cube? its volume?

51. Multiplication of polynomials. Clearly,

$$(6 + 3)(7 + 4) = 9 \times 11 = 99.$$

The multiplication may also be performed thus:

$$\begin{aligned}(6 + 3)(7 + 4) &= 6(7 + 4) + 3(7 + 4) \\ &= 42 + 24 + 21 + 12 = 99.\end{aligned}$$

In general terms,

$$(a + c)(b + d) = a(b + d) + c(b + d) = ab + ad + bc + cd.$$

EXAMPLE

Multiply $2x - 5$ by $3x + 2$.

Solution.

$$2x - 5$$

$$3x + 2$$

Multiplying by $3x$, $6x^2 - 15x$ = first partial product.

Multiplying by 2 , $+ 4x - 10$ = second partial product.

Complete product, $6x^2 - 11x - 10$ = sum of partial products.

This gives for the multiplication of polynomials the

RULE. *Multiply each term of the multiplicand by each term of the multiplier in turn, and add the partial products.*

EXERCISES

Multiply:

1. $x + 3$ by $x + 2$.

7. $2x + y$ by $x + 2y$.

2. $2x + 4$ by $x + 3$.

8. $3m + 4n$ by $3m - 5n$.

3. $3x + 6$ by $2x + 3$.

9. $3x - 2$ by $2x - 3$.

4. $2x - 5$ by $4x + 7$.

10. $-3n + 11p$ by $5n - p$.

5. $3x - 2$ by $3x + 3$.

11. $nx - px$ by $qx + rx$.

6. $5 - 4a$ by $4a - 7$.

12. $-nx + p$ by $qx - nx^2$.

13. $3x - 2y^2$ by $5x + 4y$.

14. $n^2 - 5n + 6$ by $n - 3$.

15. $3p^2 - 3p - 7$ by $2p + 4$.

16. $a^2 - ab - b^2$ by $a + b$.

17. $b^2x^2 - abx + 4b^2$ by $bx + 2b$.

18. $x^3 - 2x^2 - 5x$ by $2x^3 - 4x^2$.

19. $2r^2 - 8r + 12$ by $r^2 - 3r - 4$.

20. $x^2 - 2xy^2 - 3y$ by $x^2 - 2xy - 3y^2$.

Expand:

21. $(2x^2 - 4x - 1)(2x - 2)$.

22. $(n^2 - 3n - 2)(n^2 - 2n + 3)$.

23. $(p^2 - 3p + 2)^2$.

27. $(4m - 2m^2 - 5)^2$.

24. $(n - n^2 - 3)^2$.

28. $(3n^2 - 4n - 7)^2$.

25. $(p^2 - 2p + 4)^2$.

29. $(3n^2 - 10np + 8p^2)^2$.

26. $(3p - p^2 - 2)^2$.

30. $(2b^2c^2 - 3bc + 1)^2$.

52. Powers. A power of a number is the product obtained by using the number as a factor one or more times.

For example, 4, or 2^2 , is the second power of 2; 27, or 3^3 , is the third power of 3; $64x^6$, or $(2x)^6$, is the sixth power of $2x$.

53. Arrangement. A polynomial is said to be arranged according to the ascending powers of a certain letter when the exponents of that letter in successive terms increase from left to right.

Thus, $8 + 3x - 5x^2 + 2x^4$ is arranged according to ascending powers of x .

Again, $y^2 - 3y + 4$ and $y^4 - 3xy^3 + 3xy^2 - x^3$ are arranged in descending powers of y .

Whenever it is possible, the multiplicand and the multiplier should be arranged in the same order with respect to some letter, as the addition of the partial products is then more easily performed.

54. Check of multiplication. The work of multiplication can be checked by giving a small numeric value to each letter involved and finding the corresponding numeric values of the multiplicand, the multiplier, and the product. The product of the numeric values of the first two should equal the numeric value of the product.

The smallest number which gives a reliable check is 2. This is true when only one letter is involved. If more letters are involved, the check is not certain if 2 is used.

The number 1 is sometimes convenient for checking, but it will not check exponents, since $a^2 = a^3 = a^4 = a^{15}$, etc., if $a = 1$.

EXAMPLE

Multiply $2x^3 - 7 - 4x + 3x^2$ by $x^2 - 6 - 5x$, and check.

Solution. Arranging both terms in descending powers of x and multiplying, we obtain:

$$\begin{array}{rcl}
 \text{If} & & x = 2, \\
 2x^3 + 3x^2 - 4x - 7 & = & 16 + 12 - 8 - 7 = 13 \\
 x^2 - 5x - 6 & = & 4 - 10 - 6 = -12 \\
 \hline
 2x^5 + 3x^4 - 4x^3 - 7x^2 & & -156 \\
 -10x^4 - 15x^3 + 20x^2 + 35x & & \\
 -12x^3 - 18x^2 + 24x + 42 & & \\
 \hline
 2x^5 - 7x^4 - 31x^3 - 5x^2 + 59x + 42 & &
 \end{array}$$

Check. $64 - 112 - 248 - 20 + 118 + 42 = -156$.

Since -156 is obtained in both parts of the check, the result is probably correct.

EXERCISES

Multiply and check:

1. $2x^2 - 3x - 8$ by $2x - 4$.

2. $3m^2 - 2m + 7$ by $2m^2 - 3m + 3$.

3. $2x^2 + x - 4$ by $x^2 + x + 2$.

4. $3n^2 - 8n - 1$ by $n^2 + 2n - 3$.

5. $3x^2 - 5x - 2$ by $3x^2 - 5x - 2$.

6. $3x^3 - 4x + 2$ by itself.

7. $h^2 - kh + k^2$ by $h^2 + kh + k^2$.

8. $n^2 - n + 1$ by $n^2 - 2n + 2$.

9. $3x^3 + 4x^2 - 2x + 3$ by $x^2 - 3x + 4$.

10. $s^3 - 2s^2 + s - 3$ by $3s^2 - 2s + 1$.

11. $3n - n^3 + 6$ by $4n - 3n^2 - 5$.

HINT. Arrange both expressions in descending order.

12. $x^3 - x - 5$ by $3x^2 - 2x + 4$.

13. $3x - 2x^3 + x^2 - 8$ by $4 - x^2 - 4x$.

14. $3n^2 - 5n^3 + n + 1$ by $5 - n^2 + n$.

Expand:

15. $[5x - 2a - (2a - 5x)][5x - 2a + (2a - 5x)]$.

16. $(3a - 4a^2 + 6 + a^3)(2 + a^3 - 2a + 3a^2)$.

17. $(3x - 4 + 7x^3)(7 - 4x^2 + 2x^3 - 8x)$.

18. $(n^2 - 2np + 3p^2)(n^2 + 2np + 3p^2)$.

19. $(x^2y - y^2x)(3xy - 4x^2y)(2x^2y - xy^2)$.

20. $(n^2 + p^2 + q^2 - np - nq - pq)(n + p + q)$.

21. $(c + d + e)^2$.

22. $(2n - 3p + 4r)^2$.

24. $(p - q - r)^2$.

23. $(3a - 5b + 6)^2$.

25. $(n - 2p + 3r - 4s)^2$.

CHAPTER X

PARENTHESES IN EQUATIONS

55. Simple equations involving parentheses. In dealing with algebraic expressions involving parentheses great care must be exercised at all times. Accuracy in such work demands especial care in removing any parenthesis that is preceded by a minus sign.

EXAMPLE

Solve the equation $4(2x + 1) - (4x - 6) = 22$.

Solution. $4(2x + 1) - (4x - 6) = 22$.

Removing parentheses, $8x + 4 - 4x + 6 = 22$.

Combining, $4x + 10 = 22$.

Transposing, $4x = 12$.

Dividing by 4, $x = 3$.

Check. $4(2 \cdot 3 + 1) - (4 \cdot 3 - 6) = 22$.

Simplifying, $4 \cdot 7 - 6 = 22$,

or $22 = 22$.

EXERCISES

Solve and check:

1. $3(x + 3) = 15$. 2

6. $5(x - 8) + 7x = 8$. 4

2. $5(x - 1) = 25$. 6

7. $4(3x - 2) - 2 = 2x$. 1

3. $3(x + 6) = 12$. - 2

8. $16 + 2(4n - 7) - 12n = 0$. 2

4. $3(3 - x) + 2 = -1$. 4

9. $5x - 3(5 - x) - 9 = 0$. 3

5. $5(x - 4) + 14 = 9$. 3

10. $2(x + 1) - 4 = 3(x - 1)$. 1

11. $4(3x - 5) + 20 = 3(x + 9).$

12. $3(x + 5) + 7 = 5(8 + x).$

13. $8(x + 4) = 5(x + 8) + 4.$

14. $7(n - 3) - 2(3 - n) = 0.$

15. $9p - 3(2p - 4) = 2(5 - p) + 7.$

16. $3p - 9(3p + 4) = 3(p + 9).$

17. $7p - 12 - 2(p - 6) = 2p - 15.$

18. $5(3n - 1) - 7n = 3(n + 6) - 3.$

19. $(p - 4)(p + 8) = 7 - (2 - p)(p + 3).$

Solution. Expanding,

$$p^2 + 4p - 32 = 7 - (6 - p - p^2).$$

Removing parentheses,

$$p^2 + 4p - 32 = 7 - 6 + p + p^2.$$

Transposing and combining,

$$3p = 33.$$

Dividing by 3, $p = 11.$

Check. $(11 - 4)(11 + 8) = 7 - (2 - 11)(11 + 3).$

Simplifying, $7 \cdot 19 = 7 - (-9 \cdot 14);$

that is, $133 = 7 - (-126),$

or $133 = 133.$

20. $(n + 2)^2 - (n + 3)^2 = -17.$

21. $(n + 2)^2 - (n - 3)^2 - 10 = 0.$

22. $(3x - 6)(4x - 8) = 12x^2 - 96.$

23. $(n + 3)(n + 5) = (n + 15)(n - 10).$

24. $(p + 2)(p + 5) = (p - 4)(p - 7).$

25. $(n - 7)(6 + n) - (n - 5)(n + 6) + 16 = 0.$

26. $(x - 5)(x + 3)(x + 2) - 5 = (x^2 - x - 1)(x + 1).$

27. $(x^2 - 2x + 4)(x^2 + 2x + 4) - (x + 2) = (x^2 + 2)^2 + 20.$

ORAL EXERCISES

1. The length of a rectangle is $x + 2$ and its width is 3. What is its area? its perimeter?
2. The length of a rectangle is l and its width is w . What is its area? its perimeter?
3. A rectangle is $x + 2$ feet wide and $2x + 3$ feet long. What represents its area? its perimeter?
4. A rectangle is x feet wide and 3 feet longer than it is wide. What represents its length? its perimeter? its area?
5. One book costs a cents. What represents the cost of 3 books? 5 books? x books? $x + 2$ books? $x - 3$ books?
6. Represent the total area of two rectangles whose bases are $x + 2$ feet and $x + 3$ feet, and whose altitudes are $x - 3$ and $x + 5$ feet respectively.
7. Represent the area of a triangle whose base is x feet and whose altitude is 3 feet more than the base.
8. Represent the combined areas of two triangles, one of base $x + 2$ and altitude $x + 3$, and the other of base $2x - 1$ and altitude $2x + 3$.
9. What would represent the volume of a cube whose edge is x inches? whose edge is $x + 3$ inches?
10. A is x years old now. What will represent his age 2 years from now? 2 years ago? n years from now? n years ago?
11. A and B each have n dollars. If A gives B 10 dollars, how much will each have then? If A gives B x dollars?
12. A and B each have $x + 10$ dollars. B spends d dollars. A earns twice as much as B spends. What will represent the money they then have?

56. Problems involving parentheses. Two or more unknowns and the use of parentheses are involved in the solutions of the following problems. One unknown can always be represented by a single letter and the others by binomials involving this letter and one or more numbers. In some problems it will be necessary to inclose these binomials in parentheses and to think of them and use them as if each represented a single number.

EXAMPLE

The sum of two numbers is 47. Three times the less number is 4 less than twice the greater. What are the numbers?

Solution. Here $3 \times \text{less number} = 2 \times \text{greater number} - 4$.

Let $n = \text{the less number.}$

Then $47 - n = \text{the greater number.}$

Substituting these symbols in the foregoing statement,

$$3n = 2(47 - n) - 4.$$

Removing parentheses,

$$3n = 94 - 2n - 4.$$

Transposing and combining,

$$5n = 90.$$

Dividing by 5, $n = 18$.

Check. $47 - n = 29$.

$$3 \times 18 = 2 \times 29 - 4.$$

Multiplying, $54 = 58 - 4$.

Combining, $54 = 54$.

PROBLEMS

1. The sum of two numbers is 53. Twice the greater equals seven times the less minus 11. What are the numbers? $40-13$

2. The sum of two numbers is 30. Three times one of them minus twice the other equals 10. What are the numbers? $14-16$

3. Find three consecutive numbers whose sum is 78. $25-26-27$

4. Find three consecutive odd numbers whose sum is 87. $27-29-31$

5. The sum of three numbers is 110. The first is 12 more than the second, and the third is 2 less than twice the second. What are the numbers?

6. The length of a hall is 12 feet more than three times the width. It requires 120 feet of base board for the hall. What are the dimensions of the hall? $12-48$

7. Twice a number plus 4 equals three times the number less 19. What is the number? $24-4$

8. A father is 2 years more than five times as old as his son. The sum of their ages is 38. How old is each?

9. The perimeter of a rectangle is 224 feet. It is two fifths as wide as it is long. What are its dimensions?

10. Twice a certain odd number plus three times the next consecutive odd number is 31. What are the numbers?

11. The altitude of a triangle is 2 inches shorter than its base. What are its dimensions if its area is 2 square inches more than half that of a square whose side equals the altitude of the triangle?

12. The sum of two numbers is 15. The difference of their squares is 15. What are the numbers? $7-8$

13. The difference of the squares of two consecutive numbers is 31. What are the numbers? $14-15$

14. The difference of the squares of two consecutive even numbers is 108. What are the numbers? *20; 22*

15. The product of two consecutive even numbers is 76 less than the square of the greater number. What are the numbers? *36; 38*

16. One number exceeds another number by 4, and its square exceeds the square of the second number by 48. Find the numbers. *8; 4*

Solution. $(\text{Greater number})^2 - (\text{less number})^2 = 48.$

Let $G = \text{greater number.}$

Then $G - 4 = \text{less number,}$

and $G^2 - (G - 4)^2 = 48,$

or $G^2 - (G^2 - 8G + 16) = 48.$

Removing parentheses and solving, we obtain $G = 8.$

17. Find two numbers such that their difference is 7, and the difference of their squares is 119. *12; 5*

18. Find two numbers whose sum is 18, and the difference of whose squares is 108.

19. Find three consecutive odd integers such that the product of the first and second is 154 less than the square of the third.

20. Find four consecutive odd numbers such that the product of the first and third is 72 less than the product of the second and fourth.

21. Find three numbers such that the second is 7 more than the first, the third 3 less than the first. The sum of the first and the square of the second is 103 more than the square of the third.

22. The difference between the squares of two consecutive odd numbers is 144. What are the numbers?

23. A room is 4 feet longer than it is wide. If the length is increased by 4 feet and the width decreased by 2 feet, the floor area will be increased by 8 square feet. How large is the room? $(4+4)(4-2) = 7(4-2) + 8$

24. The length of a lot exceeds its width by 15 feet. If each were increased by 5, the area would be increased by 700 square feet. Find the dimensions of the lot.

25. Find a number such that if 6, 15, and 25 are in turn added to it, the product of the first and third sums will be 470 more than the product of the number and the second sum.

26. Divide \$253 among eight men, four women, and five children, so that each woman shall receive three times as much as any child, and each man \$6 more than any woman.

27. The value of 11 pieces of money, consisting of nickels and dimes, is 75 cents. How many of each are there?

HINT. Value of the nickels + value of the dimes = 75 cents.

Let n = number of nickels.

Then $11 - n$ = number of dimes.

Now $5n$ = value of the nickels in cents,

and $10(11 - n)$ = value of the dimes in cents.

Therefore $5n + 10(11 - n) = 75$.

28. The value of 23 coins, consisting of nickels and dimes, is \$1.60. How many of each are there?

29. The value of 37 coins, consisting of nickels, dimes, and quarters, is \$4.25. There are twice as many dimes as nickels. How many coins of each kind are there?

30. A boy has 53 coins, consisting of nickels, dimes, and quarters, amounting to \$5. He has twice as many nickels as dimes. How many of each kind has he?

31. There are 27 coins, consisting of nickels, dimes, quarters, and half-dollars, amounting to \$5.95. There are 2 more dimes than nickels and as many half-dollars as nickels. How many of each kind are there?

32. A rectangle is the same width as a square, but is 2 feet longer. The difference between their areas is 32 square feet. What are the dimensions of each?

33. A certain rectangle is 5 feet longer than it is wide. If 2 feet are taken from the width and 3 feet from the length, the resulting rectangle is 24 square feet smaller than the original rectangle. Find the dimensions of the rectangle.

34. The radius of one circle is 2 feet more than that of another circle. The area of the first circle exceeds the area of the second circle by 88 square feet. Find the radius of each circle.

35. There are 18 steps in a certain stairway. The depth (tread) of each step is twice the height (rise). How deep is each step if 9 yards of carpet are required to carpet the stairs?

36. The radius of one circle is 2 feet less than the radius of a second circle. The difference of their areas is 22 square feet. What are the diameters of the two circles?

37. The side of one square is 7 feet less than the side of a second square. If two other squares be constructed, one with a side 2 feet greater than the side of the smaller square, and the other with a side 2 feet less than the side of the larger square, the difference of their areas will be 69 square feet. What are the dimensions of the original squares?

38. A is three times as old as B, and C is 8 years older than B. The sum of their ages is 48 years. How old is each?

39. A is 5 years older than B, and C is 7 years younger than B. Five years ago the sum of their ages was 43. How old is each?

HINTS. $(A's\ age - 5) + (B's\ age - 5) + (C's\ age - 5) = 43.$

Let $b = B's\ age\ in\ years\ now.$

Then $b + 5 = A's\ age\ in\ years\ now,$

and $b - 7 = C's\ age\ in\ years\ now.$

Substituting these symbols, we have

$$b + 5 - 5 + b - 5 + b - 7 - 5 = 43.$$

Combining, $3b - 17 = 43.$

40. A is 5 years younger than B, and C is 10 years older than B. In 10 years the sum of their ages will be 110. How old is each now?

41. A is now 63 and B is 33. How many years ago was A three times as old as B?

42. A is now 38 and B is 59. How many years ago was A half as old as B?

43. The sum of the ages of A and B now is 30 years. In 9 years A's age will be three times B's age. How old is each?

44. A is four times as old as B. Five years ago he was three times as old as B will be one year from now. How old is each?

45. A grocer has 20 pounds of coffee worth 50 cents per pound. How many pounds of 35-cent coffee should he mix with it to produce a mixture worth 40 cents per pound?

46. A dealer has on hand walnuts priced at 45 cents a pound and Brazil nuts priced at 38 cents a pound. How many pounds of each should he take to make a mixture of 147 pounds which will be worth 39 cents per pound?

CHAPTER XI

DIVISION

57. Division by a monomial. Division is the process of finding one factor (the quotient) of a product of two factors (the dividend) when the other one of them (the divisor) is given.

Thus, the product 6 divided by the factor 3 gives the other factor, 2. Also $5x$ divided by 5 gives x , etc.

As in arithmetic, the indicated division may be written as a fraction and often simplified.

Thus, 2 divided by 5 = $\frac{2}{5}$; and 9 divided by 12 is $\frac{9}{12} = \frac{3}{4}$.

Similarly, $b \div c = \frac{b}{c}$; and $6a \div 8b$ is $\frac{6a}{8b} = \frac{3a}{4b}$.

By the definition of an exponent (p. 17),

$$a^3 = a \cdot a \cdot a \text{ and } a^5 = a \cdot a \cdot a \cdot a \cdot a.$$

$$\text{Then } \frac{a^5}{a^3} = \frac{\cancel{a} \cdot \cancel{a} \cdot \cancel{a} \cdot a \cdot a}{\cancel{a} \cdot \cancel{a} \cdot \cancel{a}} = \frac{a^2}{1} = a^2 = a^{5-3}.$$

$$\text{Similarly, } \frac{3^5}{3^2} = \frac{\cancel{3} \cdot \cancel{3} \cdot 3 \cdot 3 \cdot 3}{\cancel{3} \cdot \cancel{3}} = \frac{3^3}{1} = 3^3 = 3^{5-2}.$$

$$\text{Also, } \frac{nr^4}{r^2} = \frac{n \cdot \cancel{r} \cdot \cancel{r} \cdot r \cdot r}{\cancel{r} \cdot \cancel{r}} = nr^2 = nr^{4-2}.$$

$$\text{Again, } \frac{mn^3}{n^3} = \frac{m \cdot \cancel{n} \cdot \cancel{n} \cdot \cancel{n}}{\cancel{n} \cdot \cancel{n} \cdot \cancel{n}} = m = mn^{3-3}.$$

These examples illustrate the

PRINCIPLE. *The exponent of any letter in the quotient is equal to its exponent in the dividend minus its exponent in the divisor.*

This principle expressed in general terms is

$$n^a \div n^b = n^{a-b}.$$

What the above formula means when b is greater than a will be explained later when the need for such division arises. It will be seen then that this case is no exception to the principle expressed by the equation.

From what precedes we see that

$$\frac{m^5np^3}{m^3p^2} = m^{5-3}np^{3-2} = m^2np.$$

Also, $ax^2 \div x^2 = a.$

Hence a letter which has the same exponent in divisor and dividend should not appear in the quotient.

The law of signs in division (see Chapter III) may be indicated as follows:

$$+bx \text{ divided by } (+b) = +x.$$

$$+bx \text{ divided by } (-b) = -x.$$

$$-bx \text{ divided by } (+b) = -x.$$

$$-bx \text{ divided by } (-b) = +x.$$

Now $-16n \text{ divided by } 8p = -\frac{2n}{p}.$

Here the quotient is a fraction and the minus sign indicates that the fraction is negative.

Similarly, $8x \text{ divided by } (-2y) = -\frac{4x}{y},$

and $-20n^2p \text{ divided by } (-5q^4) = +\frac{4n^2p}{q^4}.$

For the division of monomials we have the

RULE. *Divide the numeric coefficient in the dividend by the numeric coefficient in the divisor, following the rule of signs for division.*

Write after this quotient all the letters of the dividend except those having the same exponent in the dividend as in the divisor, giving to each letter an exponent equal to its exponent in the dividend minus its exponent in the divisor.

If there are any letters in the divisor unlike those in the dividend, write these under the preceding result as a denominator.

ORAL EXERCISES

Divide:

(1. -22 by 2 .

2. 15 by (-3) .

3. -32 by (-4) .

4. $5a^3$ by a^2 .

5. a^5 by a^2 .

6. $-a^{12}$ by a^6 .

7. x^5 by $(-x)$.

8. $-x^8$ by $(-x^2)$.

9. np^3 by $(-np)$.

10. $-n^2p^5$ by n^2p .

11. $-n^2xp$ by xp .

12. $n^2x^3p^4$ by n^2p .

13. $-9a^6$ by $3a^3$.

14. $8x^2$ by $(-2x)$.

15. $-24x^6$ by $(-6x^4)$.

16. $-35a^2x^3$ by $7a^2x$.

17. $15ax^3$ by $(-3bx^3)$.

18. $14a^3b$ by $(-7ab)$.

19. $-27n^3p^6$ by $(-3n^3p^2)$.

20. $-27np^4$ by $(-9np^3)$.

21. $77x^6y^9$ by $(-7x^2y^8)$.

22. $18a^2c^3x^2$ by $2ax^2c$.

23. $49ax^5$ by $(-7bx^4)$.

24. $-38x^6y^6$ by $(-2x^2y^3)$.

25. $-28n^6p^{12}$ by $(-4n^4p^5)$.

26. $-45n^4xp^8$ by $9n^3xp^7$.

27. $\frac{15n^2p^4}{75np^3}$.

30. $\frac{52n^{13}p^{26}}{-13n^{13}p^{13}}$.

33. $\frac{-144a^6z^8}{-24a^4x^7}$.

28. $\frac{48n^{24}p^{56}}{-6n^6p^7}$.

31. $\frac{-11a^3b^{10}c^{12}}{77ab^2c^{10}}$.

34. $\frac{50x^3y^3z^3}{-2x^3yz}$.

29. $\frac{-17n^9p^8}{51n^7p^8}$.

32. $\frac{56n^7p^{14}r^{21}}{7n^2p^7r^7}$.

35. $\frac{64mn^2p^3}{-16n^3p^3}$.

58. Division of a polynomial by a monomial. In multiplying a polynomial by a monomial (p. 92) we multiply each term of the polynomial by the monomial. In division the steps of multiplication are reversed and each term of the polynomial is divided by the monomial.

Thus, $(nx + px) \div x = \frac{nx}{x} + \frac{px}{x} = n + p.$

Again, $\frac{9x^4 - 12x^3 + 21x^2}{-3x^2} = -3x^2 + 4x - 7.$

Therefore, for the division of a polynomial by a monomial, we have the

RULE. *Divide each term of the polynomial by the monomial and write these results in succession.*

ORAL EXERCISES

Perform the indicated division :

1. $\frac{x^3 - x^4}{x^2}.$

7. $\frac{8ax - 10a^2x^2}{-2ax}.$

2. $\frac{x^2y - x^3}{x^2}.$

8. $\frac{9ax^4 - 12a^3x^5}{-3ax^2}.$

3. $\frac{x^2 - x^4}{x^2}.$

9. $\frac{ab + ad}{a}.$

4. $\frac{6x^2 - 4x}{2x}.$

10. $\frac{25x^2y^2 + 30xy^4}{-5xy}.$

5. $\frac{9x - 18x^3}{-3x}.$

11. $\frac{16np^2 - 36n^2p^2}{4np^2}.$

6. $\frac{8a - 12a^3}{2a}.$

12. $\frac{21x^3y^4 - 28x^5y^6}{7x^2y^3}.$

$$13. \frac{8x^4y - 12x^6y^2 + 16x^8y^4}{4x^4y}.$$

$$14. \frac{rt^4 - st^3 + qt^2}{-t^2}.$$

$$15. \frac{12a^2b^2 + 9a^4b^3 - 27a^6b^4}{-3a^2b^2}.$$

$$16. \frac{16a^4b^5 - 24a^5b^6 - 32a^6b^7}{-8a^2b^3}.$$

$$17. \frac{75xyz - 45x^2yz^2 + 90x^3yz^3}{15xyz}.$$

$$18. \frac{5(x+3) + b(x+3)}{x+3}.$$

$$19. \frac{3(x+1) - 2x(x+1)}{x+1}.$$

$$20. \frac{(a+b) - 2(a+b)^2}{a+b}.$$

$$21. \frac{3a(3x+4) - 4a(3x+4)}{3x+4}.$$

$$22. \frac{(a-b)c - (a-b)d}{a-b}.$$

$$23. \frac{(x-y) - 3(x-y)}{-2(x-y)}.$$

$$24. \frac{16(3x-4)^4 - 24(3x-4)^5 - 48(3x-4)^7}{-8(3x-4)^4}.$$

$$25. \frac{-5(ac^2 - 2d)^3 + x(ac^2 - 2d)}{5(ac^2 - 2d)}.$$

$$26. \frac{(a+b)(x-y) + 7(x-y)}{x-y}.$$

59. Division of one polynomial by another. In arithmetic the work of long division is commonly arranged as follows:

	2276 Quotient	
Divisor, 346	787496 Dividend	
	692	<i>Check.</i> 2276
	954	346
	692	13656
	2629	9104
	2422	6828
	2076	787496
	2076	

It should be noted that in the above

$$\frac{\text{Dividend}}{\text{Divisor}} = \text{Quotient},$$

and $\text{Quotient} \times \text{Divisor} = \text{Dividend}.$

These relations hold in algebra as in arithmetic.

In algebra, division by a polynomial is carried out in a similar manner, as is seen in the following

EXAMPLES

1. Divide $2x^2 - 10x + 12$ by $x - 2$.

	<i>Solution</i>	<i>Check</i>
	2x - 6 Quotient	2x - 6
Divisor, $x - 2$	2x ² - 10x + 12 Dividend	x - 2
	2x ² - 4x	2x ² - 6x
	- 6x + 12	- 4x + 12
	- 6x + 12	2x ² - 10x + 12

2. Divide $16a + 12a^3 - 15 - 22a^2$ by $2a - 3$.

Solution

$$\begin{array}{r}
 6a^2 - 2a + 5 \\
 2a - 3 \overline{) 12a^3 - 22a^2 + 16a - 15} \\
 \underline{12a^3 - 18a^2} \\
 -4a^2 + 16a \\
 \underline{-4a^2 + 6a} \\
 +10a - 15 \\
 \underline{+10a - 15} \\
 0
 \end{array}$$

Check

$$\begin{array}{r}
 6a^2 - 2a + 5 \\
 2a - 3 \\
 \hline
 12a^3 - 4a^2 + 10a \\
 \underline{-18a^2 + 6a - 15} \\
 12a^3 - 22a^2 + 16a - 15
 \end{array}$$

The method of dividing one polynomial by another is expressed in the

➤ **RULE.** Arrange both the dividend and the divisor according to the descending (or ascending) powers of some common letter, called the letter of arrangement.

➤ Divide the first term of the dividend by the first term of the divisor and write the result as the first term of the quotient.

➤ Multiply the divisor by the first term of the quotient and subtract, being careful to write the terms of the remainder in the same order as those of the divisor.

➤ To find the second term of the quotient, divide the first term of the remainder by the first term of the divisor, and proceed as before until there is no remainder, or until the remainder is of lower degree in the letter of arrangement than the divisor.

➤ **CHECK.** Quotient $\times$ Divisor = Dividend.

60. Numeric check for division. The following numeric check for division is often useful.

➤ **RULE.** Assign a small numeric value to the letter or letters of the dividend, divisor, and quotient. Then divide the numeric value of the dividend by that of the divisor. The result should equal the numeric value of the quotient.

This check has certain advantages over the check stated in the general rule for the division of polynomials, as it is shorter and involves a totally different series of operations from those used in the work of the division itself. It should be noted, however, that while 1 is the easiest number to use, it does not check errors in exponents. Hence 2 or 3 are better numbers. *Any number which makes the divisor zero must never be used.*

This method of checking division is illustrated for Example 1 (p. 113), as follows:

Check. Let x have a numeric value of 4.

Then $2x^2 - 10x + 12$ will have a value of 4 (Dividend),

$x - 2$ will have a value of 2 (Divisor),

and $2x - 6$ will have a value of 2 (Quotient).

Here $4 \div 2 = 2$.

It will be noted in the example cited that the value $x = 2$ would have made the divisor equal to zero. The value $x = 3$ could have been used, but it gave a value of the dividend equal to zero, and was avoided for the sake of clearness.

EXERCISES

Divide the following and check as directed by the teacher:

1. $a^2 + 10a + 24$ by $a + 4$. 4. $2p^2 - 3p - 2$ by $2p + 1$.

2. $n^2 + n - 6$ by $n - 2$. 5. $n^2 - 2n - 15$ by $n + 3$.

3. $5n^2 + 7n + 2$ by $n + 1$. 6. $5n^2 - 22n + 8$ by $n - 4$.

7. $3x^2 + 8x - 3$ by $x + 3$.

8. $6n^2 + 19n - 7$ by $3n - 1$.

9. $3p^2 - ap - 2a^2$ by $p - a$.

10. $n^3 - 11n - 6$ by $n + 3$.

11. $4p^2 - 8np + 3n^2$ by $2p - 3n$.
12. $8n^3 - 12n^2 + 6n - 1$ by $2n - 1$.
13. $a^3 - 14a - 8$ by $a - 4$.
14. $n^3 - 11n + 6$ by $n^2 + 3n - 2$.
15. $p^3 - 15p^2 + 65p - 63$ by $p - 7$.
16. $p^3 - 11p + 6$ by $p - 3$.
17. $a^3 + 3a^2b + 3ab^2 + b^3$ by $a + b$.
18. $2n^3 - 14n^2 + 14n + 12$ by $2n - 4$.
19. $3n^3 + 28n^2 + 89n - 240$ by $3n - 5$.
20. $37p + 6p^3 - 24 - 23p^2$ by $2p - 3$.
21. $-40p - 31p^2 + 21 + p^4 + 4p^3$ by $p^2 - 3 + 7p$.
22. $n^3 + 8$ by $n + 2$.
27. $n^3 - p^3$ by $n - p$.
23. $8p^3 + 1$ by $2p + 1$.
28. $n^3 + p^3$ by $n + p$.
24. $8n^3 - 125p^3$ by $2n - 5p$.
29. $n^6 + 343p^3$ by $n^2 + 7p$.
25. $n^3 + 125p^3$ by $n + 5p$.
30. $x^4 - 16$ by $x + 2$.
26. $27n^3 + 8p^3$ by $3n + 2p$.
31. $x^4 - 16$ by $x - 2$.

61. Inexact division. In division it frequently happens that the divisor is not contained in the dividend exactly; that is, without a remainder.

Thus, 25 divided by 4 gives a quotient of 6 and a remainder of 1. This is expressed in

$$\frac{25}{4} = 6 + \frac{1}{4}, \text{ or } 6\frac{1}{4}.$$

Similarly,
$$\frac{17a}{5} = 3a + \frac{2a}{5}.$$

The result of division when there is a remainder can be expressed in general terms thus:

$$\frac{\text{Dividend}}{\text{Divisor}} = \text{Partial Quotient} + \frac{\text{Remainder}}{\text{Divisor}}.$$

EXAMPLE

Divide $6a^2 - 13a + 3$ by $3a - 2$.

$$\begin{array}{r}
 2a - 3 \quad - \quad \overset{3}{3a - 2} \\
 \text{Solution.} \quad \underline{3a - 2} \overline{) 6a^2 - 13a + 3} \\
 \quad \quad \underline{6a^2 - 4a} \\
 \quad \quad \quad - 9a + 3 \\
 \quad \quad \quad \underline{- 9a + 6} \\
 \quad \quad \quad \quad - 3
 \end{array}$$

$$\begin{aligned}
 \text{Check. } (2a - 3) \times (3a - 2) - 3 &= 6a^2 - 13a + 6 - 3. \\
 &= 6a^2 - 13a + 3.
 \end{aligned}$$

62. General check for division. (a) *When the division is exact, multiply the divisor by the quotient. The product should be the dividend.*

(b) *When there is a remainder, multiply the divisor by the partial quotient and add the remainder to the product obtained. The result should be the dividend.*

NOTE. We saw on page 2 that it is customary to represent the product of two letters by placing one after the other with no sign between them. Thus, ab means a times b . But addition, not multiplication, is implied by placing the fraction $\frac{2}{3}$ after the number 3. This practice comes down to us from the Arabs, who denoted all additions by placing the number symbols in succession without any sign of operation. The later Greeks also had the same notation.

EXERCISES

Divide the following and check as directed by the teacher :

1. $6n^2 - 13n - 6$ by $2n - 3$.
2. $19p + 12p^2 + 21$ by $4p - 3$.
3. $6n^2 + 11n - 35$ by $2n - 7$.
4. $25p^4 + 30p^2 - 7$ by $5p^2 - 7$.

5. $-2n^2 - 8 + n^3 + 4n$ by $n - 2$.

6. $n^3 - 15n^2 + 65n - 66$ by $n - 6$.

7. $p^3 - 3p^2c + 3pc^2 - c^3$ by $p - c$.

8. $10x^2 + 10x - 50x^3 - 1$ by $5x^2 - 1$.

9. $6p^3 - 18p + 12$ by $p - 1$.

10. $3p^3 - p^2 + 5p + 9$ by $p^2 - 2p + 9$.

11. $a^3 + 28a^2 - 18a - 130$ by $a - 3$.

~~12.~~ $23x^2 - 37x - 24 - 6x^3$ by $2x + 3$.

13. $n^2 + 5np - 2n + 10p$ by $n + 5p$.

14. $4n^2 + 3cn + 8an + 6ac$ by $4n + 3c$.

15. $53p + 8 - 53p^2 + 12p^3$ by $4p^2 - 7p - 1$.

16. $25p^3 - 10p^2 + 36p + 72$ by $5p - 6$.

17. $2an + 5xy - 5ny - 3ax$ by $3x - 2n$.

~~18.~~ $3n^4 + 11n^3 - 3n^2 + 17n - 4$ by $n - 3n^2 - 4$.

~~19.~~ $23n^2 + n^4 - 55n + 11n^3 - 140$ by $n^2 - 5$.

~~20.~~ $4p^3 + 1 + p^4 + 4p + 6p^2$ by $2p + p^2 + 1$.

~~21.~~ $21 + 40p - 31p^2 + p^4 - 4p^3$ by $p^2 - 3 - 7p$.

~~22.~~ $n^4 - 8n^3 + 24n^2 - 32n + 16$ by $n^2 - 4n + 4$.

23. $224n - 42n^2 + 10n^4 - 27n^3 - 419$ by $9 + 2n^2 - 5n$.

24. $x^5 - 5x^4 + 9x^3 - 6x^2 - x + 2$ by $x^2 - 3x + 2$.

25. $30x^4 + 11x^3 - 82x^2 - 12x + 48$ by $2x - 4 + 3x^2$.

26. $138y - 36 - 142y^3 + 56y^4 - 70y^2$ by $4y^2 - 13y + 6$.

27. $12x^5 - 30x^4 + 8x^3 + 14x^2 - 14x + 4$ by $6x^3 - 2x + 2$.

~~28.~~ $4y^3 - 16y + 2y^4 + 24 - 14y^2$ by $y^2 + 2 - 6y$.

29. $28x^4 - 90x^3y + 156x^2y^2 + 90xy^3 + 28y^4$ by $4x^2 + 10xy$.

30. $n^3 - p^3$ by $n - p$.

33. $n^5 + y^5$ by $n + y$.

31. $n^6 - 1$ by $n - 1$.

34. $n^5 + p^5$ by $n - p$.

32. $n^6 - 1$ by $n + 1$.

35. $n^5 - p^5$ by $n + p$.

36. $n^5 - p^5$ by $n - p$.

37. $p^5 - 5p^2 - 3000$ by $p - 5$.

38. $64n^6 + 27p^{12}$ by $4n^2 + 3p^4$.

39. $p^2 - 2np + n^2 - a^2$ by $p - n + a$.

40. $81p^4 - 256$ by $27p^3 + 36p^2 + 48p + 64$.

41. $6n^4 - 7n^3 - 28n^2 + 8n - 21$ by $3n^2 - 5n - 7$.

42. $p^4 + p^2n^2 + n^4$ by $p^2 + pn + n^2$.

43. $8p^4 - 50p^2 + 32$ by $4p^2 + 6p - 8$.

44. $n^5 - n^4p - np^4 + p^5$ by $n^2 - 2np + p^2$.

45. $n^3 - 3npr + p^3 + r^3$ by $n + p + r$.

REVIEW EXERCISES

Name the factors in :

1. $4a^2xyz^3$. 2. $7a^3bc^5d^3$. 3. $10x^5y^{10}z$. 4. $9x^{10}yz^5$.

What is the coefficient of x in :

5. $9abx^7$. 6. $3ay^2x$. 7. $15aw^5x^4$. 8. $10abcx^2yz$.

9. What is the coefficient of a in Exercises 5-8?10. What is the coefficient of b in Exercises 5-8? of y ? of z ?What is the exponent of x in :

11. $7ax^5$. 12. $6xy^2$. 13. $9x^3m$. 14. xac^2 .

When $x = 2$, $y = 3$, $z = 5$, find the value of :

15. $\sqrt{z^2 - y^2}$. 17. $(x + y + z)^2$. 19. $x^2 + z^5$.

16. $\sqrt[3]{y + z}$. 18. $(x + y)^2 - (z - y)^2$. 20. $\sqrt{yz + xz}$.

21. $\sqrt{z^2 + y^2 + x}$.

22. $\sqrt{3yz + x^2}$.

23. If $x = 2$ and $y = -3$, find the value of $3xy^2$; $5xy$; $2x^2y^3$; $7xy - 4x^2y$.

24. From the sum of $a^2x + ax^2 + 3ay$, $4a^2x - 2ax^2 + 5ay$, $7ax^2 - 6ay$, and $-4a^2x - 3ax^2 + 3ay$ take the sum of $5a^2x - 3ax^2 - 5ay$, $-12a^2x + 2ax^2 + 7ay$, and $7ax^2 - 3ay$.

Simplify:

$$25. 2\sqrt{x-y} - 3\sqrt{x-y} + 7\sqrt{x-y} - 6\sqrt{x-y}.$$

$$26. a(a+b) + b(a+b) - c(a+b) - 3(a+b).$$

$$27. 6(x-2y) + 3(x-2y) - 5(x-2y) - a(x-2y).$$

$$28. 2ab(w+v) + 3ab(w+v) - 7(w+v) - ab(w+v).$$

$$29. (x-y)\sqrt{ab} + 12\sqrt{ab} - 4\sqrt{ab} - (x-y)\sqrt{ab}.$$

If $A = a^2 + 4ab - 3b^2$, $B = 2ab + 4b^2 - 3a^2$, $C = b^2 + 2a^2 - 3ab$, $D = a^2 - 2b^2 + 2c^2$, find the expression for:

$$(30. A + B - C - D.$$

$$31. A - B + C - D.$$

$$32. -2A + B + C - 2D.$$

Find the values of the unknown in:

$$33. 2m + 5 = m + 7.$$

$$36. 2(b-3) = b + 1.$$

$$34. 6(n+3) = 4(n+6).$$

$$37. 2x + 5 + x = 4x - 2.$$

$$35. \frac{n+5}{2} = 3n - 10.$$

$$38. 3(x+2) = 4(x - \frac{1}{2}).$$

Which of the following equations are identities and which are equations of condition?

$$39. 3a - 7 = 2a.$$

$$40. x - 3 = 3x - 4.$$

$$41. 3(x+1) = 3x + 3.$$

$$42. 3(x-3) = 5x + x - 4 + x + 1 - 2(x+6).$$

Remove the parentheses and collect like terms:

(43. $3x - 4y - [3z - 8x - (2y - 3z) - 9x + 2z]$.)

(44. $4a - [3a - (8a + 2b) + 4] - (3b - 6)$.)

(45. $[4x + (3z - 4y)] + [4x - (3z - 4y)]$.)

Multiply and check:

(46. $a - 2 - a^3 + 2a^2$ by $2a - a^2 - 1$.)

(47. $(x^2y - 2y^2x)(xy - 5x^2y)(3x^2y - xy^2)$.)

(48. $(a - 3b + 2c)^2$.)

(49. $(a - b + 2c - 3d)^2$.)

(50. $(2x - 4y)^2 - (2x + 3y)^2$.)

(51. $[a + (2a^2 + 3)][a - (2a^2 + 3)]$.)

Solve for the unknown:

(52. $(k - 7)(4 + k) - (k - 5)(k + 8) = 0$.)

(53. $(3x - 5)(4x - 2) = 12x^2 - 172$.)

(54. $(b - 5)(2b + 3) + 21 = 2(b - 5)(b + 5)$.)

Divide:

(55. $\frac{4(a^2c - 2d)^2 + x(a^2c - 2d) - 2(a^2c - 2d)^3}{2(a^2c - 2d)}$.)

(56. $\frac{x^5 + 5x^4y + 11x^3y^2 + 4x^2y^3 - 2xy^4 + y^5}{x + y}$.)

57. A woman bequeathed \$10,500 to three charities. To the second she bequeathed two fifths more than to the first, and to the third \$60 less than twice the amount given to the first. How much did each receive?

58. A man and a boy together earn \$55. The man receives three times as much per day as the boy. How much does each receive if the boy works 10 days and the man works 4 days?

*34 = man's
104 = boy's wages in 10 days
12 = man's wages in 4 days*

59. A certain number is reduced by one half and then by three eighths of its original value. The result is 125. Find the number.

60. There are 112 bushels of corn in 2 bins. In one bin there are 17 bushels less than half as many as there are in the other. How many bushels are there in each?

61. A boy bought oranges at the rate of 3 for 2 cents and gained 50 cents by selling them 2 for 3 cents. How many oranges did he buy?

62. A cow and a sheep together cost \$125. The cow cost \$5 more than three times as much as the sheep. Find the cost of each.

63. An estate of \$97,800 was left to three men. The second received \$1200 more than the first, and the third \$600 less than the first and the second together. How much did each receive?

64. Three men have together \$2000. The first has \$275 more than the second, and the second has \$300 more than the third. How much has each?

65. Three men receive a certain sum of money. The first receives twice as much as the second, the second twice as much as the third. The difference between the amounts received by the first and third is \$7500. How much does each receive?

66. A stick 117 inches long is to be cut so that the first piece is 2 inches more than twice as long as the second, and the second 5 inches less than three times as long as the third. Find the length of each piece.

67. Four boys together sold 100 papers. The first sold twice as many as the second, the third 5 more than three times as many as the second, and the fourth 4 less than the first and second together. How many papers did each sell?

$$22 - 11 = 38.33$$

$$F = \frac{9}{5} C + 32$$

$$\frac{9}{5} C + 32 = F \quad \frac{9}{5} C = F - 32$$

~~68.~~ Find three consecutive numbers such that the product of the first and third exceeds the product of the first and second by 27.

~~69.~~ Find three consecutive numbers whose sum is 84.

70. Find three consecutive even numbers such that the product of the second and third less the product of the first and third is 16 less than three times the first.

71. Find three consecutive odd numbers such that the product of the second and third exceeds the product of the first and second by 220.

CHAPTER XII

EQUATIONS AND PROBLEMS

63. Equations involving literal coefficients. In the most general form of a simple equation in one unknown the constant term is represented by a letter and the unknown has literal coefficients. The simplest general form is $ax = b$, in which x is the unknown and a and b are known numbers. The solution of such equations involves no new principle. It is merely necessary to perform the usual operations with letters instead of with arithmetic numbers.

In solving equations whose coefficients involve letters the answers obtained will usually involve these same letters. Only in exceptional cases may one expect to obtain the root of such an equation in purely numeric form.

In the following exercises the unknown is represented by a letter near the end of the alphabet, as x or y , and the letters in the coefficients and terms supposed to be known are taken from the other letters of the alphabet.

ORAL EXERCISES

Solve for x , y , v , or z :

- | | | |
|--------------------|-----------------------|---------------------|
| 1. $x - c = 0$. | 4. $x - 7m = 2m$. | 7. $az + ab = ac$. |
| 2. $x + 3b = 8b$. | 5. $x - 3m = 1 - m$. | 8. $nz - an = bn$. |
| 3. $x + b = c$. | 6. $3z - 6b = 9m$. | 9. $v - 6a = 12a$. |

10. $5v = 10m.$ 16. $3cy = 9c - 6c^3.$
 11. $2av = 6a^2.$ 17. $4ay - 2a^2 = 6a^2.$
 12. $mv - 3m^2 = m^2.$ 18. $3cy + c^2 = 7c^2.$
 13. $av = 4a^3.$ 19. $(m+n)x = m+n.$
 14. $ay = ab + a^2.$ 20. $(m-n)y = (m-n)^2.$
 15. $2y = 4m - n.$ 21. $(m+n)z = 2a(m+n).$
 22. $(n+t)(n-t)x = (n+t)(n-t).$
 23. $(n-t)(n+t)y = 2c(n+t)(n-t).$

EXERCISES

Solve for the unknown, and check:

1. $5x - 2a = 10a + 3x.$

Solution. $5x - 2a = 10a + 3x.$

Transposing, $5x - 3x = 10a + 2a.$

Combining, $2x = 12a.$

Dividing by 2, $x = 6a.$

Check. Substituting $6a$ for x in the first equation,

$$5(6a) - 2a = 10a + 3(6a)$$

$$30a - 2a = 10a + 18a, \text{ or } 28a = 28a.$$

2. $mx + m = 5m.$ $\times$ 10. $5ax - 10a^2 = 5ac.$

3. $x - m = m + n.$ 11. $am - mx = 3am.$

4. $x - c = m - c.$ 12. $3(a+x) = 6a.$

$\times$ 5. $3mx - 5m = 7m.$ $\times$ 13. $4(a-x) = 10a.$

6. $ay - am = ac.$ $\times$ 14. $12c - 3(c-x) = 0.$

7. $my - m^2 = 5m^2.$ 15. $ay - (a-c) = 3a+c.$

8. $3bx - b^2 = 8b^2.$ 16. $3ay - ab = 2ay - ac.$

9. $mn + nx = 5mn.$ $\times$ 17. $4nz - 7n^2m = 3nz + 6mn^2.$

$$18. 3ax + 2ab = 6ab + 2ax - 3ab.$$

$$19. mx + n^2 = 4m^2 - (mx - n^2).$$

$$20. ax + 5a = a^2 + 6 + 3x.$$

Solution. $ax + 5a = a^2 + 6 + 3x.$

Transposing, $ax - 3x = a^2 - 5a + 6.$

Uniting the coefficients of x ,

$$(a - 3)x = a^2 - 5a + 6.$$

Dividing both members by $a - 3$,

$$x = \frac{a^2 - 5a + 6}{a - 3}.$$

Performing the division,

$$x = a - 2.$$

Check. Substituting $a - 2$ for x in the original equation,

$$a(a - 2) + 5a = a^2 + 6 + 3(a - 2).$$

Simplifying, $a^2 - 2a + 5a = a^2 + 6 + 3a - 6,$

or

$$a^2 + 3a = a^2 + 3a.$$

$$21. ax + bc = bx + ac.$$

$$23. ay + 2ab = 2a^2 + by.$$

$$22. my + 1 - m^2 - y = 0.$$

$$24. nx + 1 - n^3 - x = 0.$$

$$25. ax - 2a^3 - 1 = a^2 - x.$$

$$26. x - 5n^2 = 2n^3 - 2nx - 1.$$

$$27. 6mc + ax + an = 3am + 2cx + 2cn.$$

$$28. 2a(a + c) + 3nx = a(3n + 2x) + 3nc.$$

HINT. Remove parentheses first.

$$29. (y - a)(y + n) + an + a^2 = y^2 + n^2.$$

$$30. 15(y - n) - 6(y + n) = 3(5n - 3y).$$

$$31. 4x - cx - 8 - 4a + 6c = 2cx - 3ac.$$

$$32. 2ny - 2n = 1 - y.$$

64. Uniform motion. If a man rides for 8 hours at the uniform (unvarying) speed of 20 miles per hour, he will travel in all 8×20 , or 160, miles. This example illustrates *uniform motion*. In all problems of uniform motion the elements involved are

(a) Time, measured in seconds, minutes, hours, etc.

(b) Rate of motion (speed), or the distance traveled in a unit of time (one second, one hour, one day, as the case may be).

(c) Distance (total), measured in feet, inches, miles, meters, etc.

If a body moves at a constant speed r for a time t through a distance d , then d , r , and t are connected by the equation

$$d = r \times t. \quad (1)$$

This relation or formula gives an insight into the power of the algebraic language. By means of arithmetic numbers alone we can express the relation which holds between the time, the speed, and the distance only for a particular case. By means of the literal equation (1) we express the relation which is true not merely for one case but for countless cases. In fact, it holds whenever we are dealing with uniform motion, whatever numeric values d , r , and t may have.

ORAL EXERCISES

1. An automobile ran 35 miles per hour for 7 hours. How far did it run?

2. An automobile runs 320 miles in 8 hours. Find the average speed.

3. A steamship travels 1920 miles in 4 days. What is its speed in miles per hour?

4. How far does an automobile travel if it runs

- (a) 25 miles per hour for 6 hours?
- (b) 20 miles per hour for h hours?
- (c) m miles per hour for h hours?
- (d) 30 miles per hour for $t + 5$ hours?
- (e) $2x - 4$ miles per hour for 7 hours?

Sound travels about 1100 feet per second. Light travels 186,300 miles per second. Therefore the interval of time between the occurrence and the observation of an event may be ignored in the following problems:

5. An observer hears the stroke of an ax $1\frac{2}{5}$ seconds after he sees the ax strike the tree. How far is the observer from the woodcutter?

6. How far is an observer from a rocket if he hears it $7\frac{1}{2}$ seconds after he sees the flash of the explosion?

7. What is the speed of an automobile if it runs uniformly

- (a) 240 miles in 8 hours?
- (b) d miles in 8 hours?
- (c) d miles in h hours?
- (d) 240 miles in $x + 3$ hours?
- (e) d miles in $t - 2$ hours?

8. The United States Air Mail Service has a schedule of 32 hours from New York to San Francisco, a distance of 2620 miles. What is the average speed in miles per hour?

9. An automobile runs a distance d at a speed of 25 miles per hour. Another car runs 90 miles farther at 30 miles per hour. Represent the number of hours required by each car for its run. If each requires the same time, what equation can be stated?

10. An automobile runs 25 miles per hour for t hours, a second one runs 18 miles per hour for $t + 3$ hours. Represent the distance each travels. If the two cars travel the same distance, what equation can be stated?

11. If the first automobile in Exercise 10 runs 20 miles farther than the second, what equations can be stated?

12. Two automobiles leave two towns 450 miles apart at the same time and travel toward each other, one at the rate of t miles per hour, the other twice as fast. They meet in 10 hours. Represent the distance each has traveled during that time. What equation involving t exists?

13. An aëroplane leaves one city to fly to another 720 miles away. An hour later another aëroplane leaves the second city to fly to the first. The two meet midway between the cities. Represent the speed of each. If one travels 30 miles per hour faster than the other, state the equation involving t which represents the conditions of the problem.

EXAMPLES

1. A motorist traveling 20 miles per hour is overtaken in 12 hours after leaving a certain point by a second motorist, who left the same starting point 4 hours after the first. Find the speed of the second motorist.

Solution. This is a problem in uniform motion, involving the distance, the speed, and the time of the first and of the second motorist. By a careful reading of the problem one discovers that the time of each was a different number of hours, that each went at a different speed, but that each traveled the same distance. Hence an equation can be formed by expressing d in terms of r and t for each and then equating the two expressions for d .

By the conditions of the problem :

	t , OR TIME IN HOURS	r , OR RATE IN MILES PER HOUR	DISTANCE IN MILES $r \times t = d$
First motorist	12	20	$20 \times 12 = 240$
Second motorist	8	x	$8x$

and $8x = 240$,

then $x = 30$.

Check. $20 \times 12 = 240$. $30 \times 8 = 240$.

2. Two men, A and B, start from the same place at the same time and travel in opposite directions. B goes three times as fast as A. In 6 hours they are 96 miles apart. Find the speed of each.

Solution. By the conditions of the problem :

	t , OR TIME IN HOURS	r , OR RATE IN MILES PER HOUR	DISTANCE IN MILES $r \times t = d$
A	6	r	$6r$
B	6	$3r$	$18r$

and $6r + 18r = 96$;

then $24r = 96$,

whence $r = 4$ and $3r = 12$.

Check. $12 = 3 \cdot 4$; $6 \cdot 4 + 6 \cdot 12 = 96$.

It should be particularly noted in choosing letters for the unknowns that it is not enough to say, for instance, let x equal the distance or let t equal the time. This means little unless the unit of distance or the unit of time is stated. The unknown distance is a number of feet or miles or some other unit of length, and the unknown time is a





Sir Isaac Newton

number of seconds or hours or some other specific unit of time. This should be borne in mind each time a letter is taken to represent the measure of any physical quantity.

BIOGRAPHICAL NOTE

SIR ISAAC NEWTON. Sir Isaac Newton (1642–1727) was probably the keenest mathematical thinker who ever lived. He was the son of a farmer of slender means, and as a boy was rather lazy. It is said, however, that his complete victory over a larger boy in a fight at school led him to feel that perhaps he could be equally successful in his studies if he really tried. His ambition and interest being once aroused, he never ceased to apply himself during the rest of his long life.

His most important scientific achievement was the discovery and verification of the laws of motion. In his great work called the "Principia" he showed by mathematical reasoning that all bodies, great and small, — the planet revolving around the sun, as well as the apple falling from the tree, — follow the same laws. His greatest discovery in pure mathematics was that of a method called the calculus, which is the basis of most of the advances in mathematics and in theoretical physics made since his time.

But important as was Newton's mathematical work, his most significant contribution to mankind was an idea, — the idea that the world in which we live is not independent of the rest of the universe, but that every smallest particle of matter is connected with the most remote planet and star; that we cannot think of the earth as the center of all things, but that we merely occupy our place in a system governed by universal law.

MISCELLANEOUS PROBLEMS

In Problems 1–5, A and B start at the same time from two points 168 miles apart and travel toward each other. Find the speed of each:

1. If they travel at the same speed and meet in 7 hours.
2. If A travels 2 miles per hour faster than B and they meet in 12 hours.

3. If B travels three times as fast as A and they meet in 7 hours.

4. If A travels 48 miles farther than B and they meet in 6 hours.

5. If they meet in 14 hours and A travels 2 miles per hour less than B.

6. C and D travel the same distance, C in 8 hours, D in 10. C's speed is 3 miles per hour more than D's. Find the speed of each and the distance traveled.

7. The first transatlantic non-stop flight was made by Alcock and Brown from St. John's, Newfoundland, to Clifden, Ireland, in June, 1919. Their speed was $121\frac{1}{9}$ miles per hour. Had it been $21\frac{1}{9}$ miles per hour less, the time taken would have been increased 3.42 hours. Find the distance flown.

In Problems 8–11, A and B start at the same time from two points 252 miles apart and travel toward each other until they meet. Find the number of hours from the start until the time of meeting:

8. If A travels 6 miles per hour faster than B and twice as far.

9. If A travels 5 miles more per hour than B and also travels three times as far.

10. If B travels 9 miles per hour and A travels 12 miles per hour but is delayed 2 hours on the way.

11. If B is delayed 2 hours and A is delayed 6 hours and their rates while moving are 15 and 17 miles per hour respectively.

12. A starts from a certain place and travels 20 miles per hour. Two hours later B starts from the same place and travels in the same direction at the rate of 25 miles per hour. How long after A starts will B overtake A?

13. On May 2, 1923, Lieutenants Kelly and Macready of the United States Air Service with the plane T-2 broke all previous records by flying from New York to San Diego in 26 hours and 51 minutes. Had they flown 167 miles farther in the same time their rate would have been 100 miles per hour. Find their average speed and the distance from New York to San Diego.

14. A traveler having 4 hours at his disposal wishes to ride out of town on a trolley car whose speed is 12 miles per hour and return on foot at a rate of 3 miles per hour. On the way back his route is 2 miles longer than that of the car going out. How long and how far may he ride?

The velocity of a bullet continually decreases from the instant it leaves the gun. This is owing to the resistance of the air. In the following problems consider the velocity of sound to be 1100 feet per second:

15. A marksman hears the thud of a bullet striking a target 880 yards away 4 seconds after he presses the trigger. Find the average velocity of the bullet.

16. Two seconds after a marksman presses the trigger of his revolver he hears the bullet strike the target 60 rods away. Find the average velocity of the bullet.

17. The distance from New York to Chicago is 912 miles. The Manhattan Limited makes the run in 22 hours. An eastbound train leaves Chicago at the same time the Manhattan Limited leaves New York. The two pass each other twelve hours later. Find the speed of each train.

CHAPTER XIII

IMPORTANT SPECIAL PRODUCTS

65. The square of a binomial. The multiplication

$$\begin{array}{r} a + b \\ a + b \\ \hline a^2 + ab \\ + ab + b^2 \\ \hline a^2 + 2ab + b^2 \end{array}$$

gives the formula $(a + b)^2 = a^2 + 2ab + b^2$.

This may be expressed in words as follows :

I. The square of the sum of two terms is the square of the first term, plus twice the product of the two terms, plus the square of the second term.

Similarly, $(a - b)^2 = a^2 - 2ab + b^2$,

which may be expressed in words as follows :

II. The square of the difference of two terms is the square of the first term, minus twice the product of the two terms, plus the square of the second term.

Study the application of I and II in the following

EXAMPLES

1. $(a + 1)^2 = a^2 + 2a + 1$. 3. $(4 + x)^2 = 16 + 8x + x^2$.

2. $(a - 3)^2 = a^2 - 6a + 9$. 4. $(3 - y)^2 = 9 - 6y + y^2$.

5. $(3x + a)^2 = 9x^2 + 6ax + a^2$.

ORAL EXERCISES

Expand the following by I or II of page 134:

- | | | |
|-------------------|--------------------|----------------------|
| 1. $(x + 1)^2$. | 9. $(10 - y)^2$. | 17. $(7x - t)^2$. |
| 2. $(a + 2)^2$. | 10. $(a + t)^2$. | 18. $(11m + a)^2$. |
| 3. $(x - 5)^2$. | 11. $(x - t)^2$. | 19. $(x + .2)^2$. |
| 4. $(y + 7)^2$. | 12. $(m - y)^2$. | 20. $(t - .3)^2$. |
| 5. $(2 - x)^2$. | 13. $(x - 2t)^2$. | 21. $(3t + .4)^2$. |
| 6. $(10 - a)^2$. | 14. $(3t - a)^2$. | 22. $(t + .05)^2$. |
| 7. $(4 + x)^2$. | 15. $(5x + m)^2$. | 23. $(2t - .09)^2$. |
| 8. $(a - 6)^2$. | 16. $(3a - y)^2$. | 24. $(.5 - 4r)^2$. |

State two equal binomials whose product is:

- | | | |
|----------------------|------------------------|------------------------|
| 25. $x^2 + 2x + 1$. | 27. $a^2 - 4a + 4$. | 29. $t^2 - 14t + 49$. |
| 26. $x^2 + 6x + 9$. | 28. $t^2 - 10t + 25$. | 30. $36 - 12x + x^2$. |

Expand and find the value of:

31. $(8 + 3)^2$.

Solution. $(8 + 3)^2 = 8^2 + 2 \cdot 8 \cdot 3 + 3^2 = 64 + 48 + 9 = 121$.

32. $(7 + 1)^2$. 33. $(9 + 3)^2$. 34. $(11 + 2)^2$.

35. $(11 - 5)^2 = (11)^2 - 2 \cdot 11 \cdot 5 + 5^2 = 121 - 110 + 25 = 36$.

36. $(10 - 3)^2$. 39. $(21)^2$.

37. $(11 - 4)^2$. HINT. $(21)^2 = (20 + 1)^2$, etc.

38. $(12 - 3)^2$. 40. $(31)^2$.

41. $(22)^2$. 43. $(52)^2$. 45. $(71)^2$. 47. $(92)^2$.

42. $(41)^2$. 44. $(63)^2$. 46. $(85)^2$. 48. $(101)^2$.

49. $(19)^2$. 50. $(38)^2$. 52. $(78)^2$.

HINT. $(19)^2 = (20 - 1)^2$, etc. 51. $(29)^2$. 53. $(199)^2$.

EXERCISES

Expand the following :

1. $(2x + 3)^2$. 7. $(8a + 11b)^2$. 13. $(5xy - 3xz)^2$.
 2. $(3x - 5)^2$. 8. $(3x^2 + 8t)^2$. 14. $(10x^2y - 7xy^2)^2$.
 3. $(2t - 4m)^2$. 9. $(3x^2 - 5x)^2$. 15. $(.5a + 4t)^2$.
 4. $(4y - 3t)^2$. 10. $(4x^3 - 7x)^2$. 16. $(.4ax + 6a)^2$.
 5. $(10x - 3y)^2$. 11. $(3ax + 4at)^2$. 17. $(.4t - .8x)^2$.
 6. $(7a + 6b)^2$. 12. $(-2a + 3bx)^2$. 18. $(10x - .3t)^2$.

What changes must be made in the following so that the numerator will be the square of the denominator?

19. $\frac{a^2 + ab + b^2}{a + b}$. 27. $\frac{9x^2 - 6x + 4}{3x - 2}$.
 20. $\frac{a^2 - at + t^2}{a - t}$. 28. $\frac{x^2 - 20x + 25}{2x - 5}$.
 21. $\frac{a^2 + 10a + 100}{a + 10}$. 29. $\frac{x^2 - 24x + 9}{4x - 3}$.
 22. $\frac{x^2 + 6x - 9}{x + 3}$. 30. $\frac{4a^2 + 20at + 25t^2}{5t - 2a}$.
 23. $\frac{x^2 + 18x + 81}{x + 9}$. 31. $\frac{9x^2 - 12ax + a^2}{2a - 3x}$.
 24. $\frac{t^2 - 4t + 16}{t - 4}$. 32. $\frac{16x^2 - 24ax + a^2}{3a - 4x}$.
 25. $\frac{a^4 - 20a^2 + 25}{a^2 - 5}$. 33. $\frac{x^2 + .8x + .16}{x - .4}$.
 26. $\frac{a^2 - 7a + 49}{a + 7}$. 34. $\frac{4t^2 - .4t + .01}{2t + .1}$.

66. The product of the sum and the difference of two terms.

The multiplication

$$\begin{array}{r} a + b \\ a - b \\ \hline a^2 + ab \\ - ab - b^2 \\ \hline a^2 - b^2 \end{array}$$

gives the formula $(a + b)(a - b) = a^2 - b^2$.

This may be expressed in words as follows:

III. The product of the sum and the difference of two terms, equals the difference of their squares, taken in the same order as the difference of the terms.

Study the application of III in the following

EXAMPLES

1. $(a + 2)(a - 2) = a^2 - 4$.
2. $(7 - x)(7 + x) = 49 - x^2$.
3. $(m + t)(m - t) = m^2 - t^2$.
4. $(4a - t)(4a + t) = 16a^2 - t^2$.
5. $(2x^2 + 7)(2x^2 - 7) = 4x^4 - 49$.

ORAL EXERCISES

Perform the indicated operation:

1. $(x + 3)(x - 3)$.
2. $(a + 5)(a - 5)$.
3. $(t - 6)(t + 6)$.
4. $(1 + a)(1 - a)$.
5. $(t + x)(t - x)$.
6. $(1 + 3x)(1 - 3x)$.
7. $(x + 5t)(x - 5t)$.
8. $(4a - t)(4a + t)$.
9. $(7x + a)(7x - a)$.
10. $(11a - t)(11a + t)$.
11. $(a^2 + 12)(a^2 - 12)$.
12. $(4x - 3y)(4x + 3y)$.
13. $(2a + 7t)(2a - 7t)$.
14. $(4am - 3)(4am + 3)$.
15. $(ax + 7)(ax - 7)$.
16. $(a^2 + 5)(a^2 - 5)$.
17. $(a^4 + 12)(a^4 - 12)$.
18. $(xy - y^2)(xy + y^2)$.

19. $(8at - t^2)(8at + t^2)$. 22. $(x^2 - 4) \div (x - 2)$.
 20. $(.5x - t)(.5x + t)$. 23. $(9x^2 - 1) \div (3x + 1)$.
 21. $(9a - .7t)(9a + .7t)$. 24. $(49a^4 - 36) \div (6 + 7a^2)$.

State two binomials whose product is:

25. $a^2 - 4$. 31. $.01 - t^2$. 37. $100a^2 - 1$.
 26. $t^2 - 25$. 32. $a^2 - .09$. 38. $64a^2x^2 - 25$.
 27. $36 - x^2$. 33. $m^2 - .0016$. 39. $1 - 36m^2$.
 28. $y^2 - 64$. 34. $.16a^2 - 25$. 40. $25 - 49a^2$.
 29. $49 - y^4$. 35. $x^2 - 9t^2$. 41. $100m^2 - 16$.
 30. $81 - x^2$. 36. $16a^2 - 25y^2$. 42. $36x^2y^4 - z^2$.

Find the value of:

43. $(11 + 3)(11 - 3)$. 47. $21 \cdot 19$.
 44. $(20 - 1)(20 + 1)$. HINT. $21 \cdot 19 = (20 + 1)(20 - 1)$, etc.
 45. $(30 + 2)(30 - 2)$. 48. $22 \cdot 18$.
 46. $(50 - 1)(50 + 1)$. 49. $29 \cdot 31$.

EXERCISES

Perform the indicated operations:

1. $[x + y + t][x + y - t]$.

Solution. This is similar to $(a + b)(a - b) = a^2 - b^2$ if we regard $x + y$ as replacing a , and t as replacing b . Inclosing $x + y$ in parentheses to indicate this we have

$$\begin{aligned} [x + y + t][x + y - t] &= [(x + y) + t][(x + y) - t] \\ &= (x + y)^2 - t^2 \\ &= x^2 + 2xy + y^2 - t^2. \end{aligned}$$

2. $[(x + y) + 2][(x + y) - 2]$.

3. $[x + 3 - a][x + 3 + a]$.

4. $[x + 4 + c][x + 4 - c]$.

$$5. [(2a - b) + c][(2a - b) - c].$$

$$6. [3x - t - 5c][3x - t + 5c].$$

$$7. [2t + a - 4x][2t + a + 4x].$$

$$8. [5a - b + 2c][5a - b - 2c].$$

$$9. [a + x + y][a - x - y].$$

Solution. $[a + x + y][a - x - y] = [a + (x + y)][a - (x + y)]$
 $= a^2 - (x + y)^2$
 $= a^2 - (x^2 + 2xy + y^2)$
 $= a^2 - x^2 - 2xy - y^2.$

$$10. [t + x + a][t - x - a]. \quad 12. [t + a - x][t - a + x].$$

$$11. [t + a + 3][t - a - 3]. \quad \text{HINT. This may be written } [t + (a - x)][t - (a - x)], \text{ etc.}$$

$$13. [x - a + 2][x + a - 2].$$

$$14. [3 + x + y][3 - x - y].$$

$$15. [4a + 2y - x][4a - 2y + x].$$

$$16. [10 - t + 5s][10 + t - 5s].$$

67. The product of two binomials having a common term.
 The multiplication

$$\begin{array}{r} x + a \\ x + b \\ \hline x^2 + ax \\ + bx + ab \\ \hline x^2 + (a + b)x + ab \end{array}$$

gives the formula

$$(x + a)(x + b) = x^2 + (a + b)x + ab.$$

This may be expressed in words as follows:

IV. *The product of two binomials having a common term equals the square of the common term, plus the algebraic sum of the unlike terms multiplied by the common term, plus the algebraic product of the unlike terms.*

Study the application of IV in the following

EXAMPLES

1. $(x + 2)(x + 3) = x^2 + (2 + 3)x + 6 = x^2 + 5x + 6.$
2. $(x - 5)(x - 3) = x^2 + (-5 - 3)x + 15 = x^2 - 8x + 15.$
3. $(x + 4)(x - 7) = x^2 + (4 - 7)x - 28 = x^2 - 3x - 28.$

ORAL EXERCISES

Perform the indicated operations:

- | | |
|--|--|
| 1. $(a + 1)(a + 3).$ | 16. $(x - 6)(x + 3).$ |
| 2. $(a + 3)(a + 4).$ | 17. $(x + 4)(x - 5).$ |
| 3. $(a + 2)(a + 4).$ | 18. $(x - 8)(x + 4).$ |
| 4. $(x + 5)(x + 4).$ | 19. $(t + a)(t + b).$ |
| 5. $(x + 7)(x + 3).$ | 20. $(t + m)(t - a). t^2 + tm + \dots$ |
| 6. $(x + 5)(x + 6).$ | 21. $(x - b)(x + s).$ |
| 7. $(t + 4)(t + 7).$ | 22. $(x^2 + 3x + 2) \div (x + 2).$ |
| 8. $(a - 1)(a - 4).$ | 23. $(x^2 + 5x + 6) \div (x + 2).$ |
| 9. $(a - 2)(a - 5).$ | 24. $(x^2 + 7x + 10) \div (x + 5).$ |
| 10. $(a - 3)(a - 4).$ | 25. $(a^2 - 6a + 8) \div (a - 4)$ |
| 11. $(a - 5)(a - 3).$ | 26. $(a^2 - a - 6) \div (a - 3).$ |
| 12. $(a - 6)(a - 5).$ | 27. $(a^2 - a - 12) \div (a - 4).$ |
| 13. $(a - 5)(a - 7).$ | 28. $(a^2 + a - 20) \div (a + 5).$ |
| 14. $(x - 7)(x - 8).$ | 29. $(a^2 + a - 42) \div (a - 6).$ |
| 15. $(x - 5)(x + 2).$ | 30. $(t^2 - at - 6a^2) \div (t - 3a).$ |
| 31. $(a^2 + 4at - 21t^2) \div (a - 3t).$ | |
| 32. $(x^2 + 7xy - 18y^2) \div (x + 9y).$ | |
| 33. $(t^2 - at - 90a^2) \div (t - 10a).$ | |

Supply the missing term or sign and then carry out the following as exact divisions:

34. $(t^2 - t - 30) \div (t + ?)$.

39. $(x^2 - 3x - 40) \div (x ? 5)$.

35. $(t^2 + t - 56) \div (t + ?)$.

40. $(x^2 + 3x - 28) \div (x ? 4)$.

36. $(a^2 - 4a - 32) \div (a - ?)$.

41. $(x^2 - 5x - 24) \div (x + ?)$.

37. $(a^2 + 2a - 48) \div (a - ?)$.

42. $(x^2 + 3x - 10) \div (x ? 5)$.

38. $(a^2 - a - 72) \div (a ? 9)$.

43. $(x^2 - x - 42) \div (x - ?)$.

68. The product of two binomials of the form $(ax + b)(cx + d)$. The multiplications

$$\begin{array}{r} \cancel{2x} + \cancel{5} \\ \cancel{3x} - \cancel{7} \\ \hline \end{array}$$

$$6x^2 + 15x$$

$$- 14x - 35 \quad \text{and}$$

$$6x^2 + \quad x - 35$$

$$\begin{array}{r} \cancel{ax} + \cancel{b} \\ \cancel{cx} + \cancel{d} \\ \hline \end{array}$$

$$acx^2 + bcx$$

$$+ adx + bd$$

$$acx^2 + (bc + ad)x + bd$$

show that in the product of any two binomials of the form $(ax + b)(cx + d)$

(a) the first term is the product of the first terms of the binomials,

(b) the second term is the sum of the cross products, and

(c) the third term is the algebraic product of the second terms of the binomials.

EXAMPLES

1. $(\underbrace{2x + 3})(\underbrace{3x + 4}) = 6x^2 + 9x + 8x + 12 = 6x^2 + 17x + 12.$

2. $(\underbrace{3t + 5})(\underbrace{4t - 1}) = 12t^2 + 20t - 3t - 5 = 12t^2 + 17t - 5.$

3. $(\underbrace{5a + 2})(\underbrace{2a - 7}) = 10a^2 - 31a - 14.$

EXERCISES

Write the products of the following :

- | | |
|-------------------------|----------------------------|
| 1. $(2a + 3)(2a + 1)$. | 12. $(4x + 3)(x - 5)$. |
| 2. $(2a + 2)(2a + 3)$. | 13. $(4t - 5)(t - 3)$. |
| 3. $(3t - 1)(3t + 4)$. | 14. $(4t - 1)(t + 5)$. |
| 4. $(2t + 3)(2t - 5)$. | 15. $(3a - 2)(a - 4)$. |
| 5. $(2t - 3)(2t + 5)$. | 16. $(2a - 1)(3a + 2)$. |
| 6. $(2x - 3)(2x + 4)$. | 17. $(2x + 3)(3x - 1)$. |
| 7. $(2x - 1)(2x + 5)$. | 18. $(3x - 2)(4x + 3)$. |
| 8. $(2a - 3)(a - 1)$. | 19. $(5t - 6x)(4t + 5x)$. |
| 9. $(2a - 2)(a - 3)$. | 20. $(4a - 3t)(5a - 7t)$. |
| 10. $(3t + 4)(t - 1)$. | 21. $(5x - 2a)(7x + 3a)$. |
| 11. $(3a - 5)(a + 2)$. | 22. $(4x - 9t)(2x - 5t)$. |

ORAL REVIEW EXERCISES

Perform the following indicated operations :

- | | |
|----------------------------|---|
| 1. $(h + 2k)^2$. | 12. $(5n^3 - 4n^2h)^2$. |
| 2. $(3h - k)^2$. | 13. $(7n^2 - h)(7n^2 + h)$. |
| 3. $(n - 4h)^2$. | 14. $(-3n^2h - 2h^2)^2$. |
| 4. $(2m + 3k)^2$. | 15. $(29)^2 = 741$. |
| 5. $(m - 4r)^2$. | 16. $(71)^2 = 5041$. |
| 6. $(m^2 + 3r)^2$. | 17. $(89)^2 = 7921$. |
| 7. $(n - 3r)(n + 3r)$. | 18. $(n + 7)(n + 3) = n^2 + 10n + 21$. |
| 8. $(h - 4n)(h + 4n)$. | 19. $(n - 8)(n - 3)$. |
| 9. $(3n^2h + 2h)^2$. | 20. $(n - 6)(n + 9)$. |
| 10. $(5m^2 - k^2)^2$. | 21. $(7n^2r - 3)(7n^2r + 3)$. |
| 11. $(3n - 5h)(3n + 5h)$. | 22. $(t^2 + 16)(t^2 - 16)$. |

$$23. (4t^2 - 9)(4t^2 + 9).$$

$$24. (n - t)(n^2 + t^2)(n + t).$$

HINT. Multiply the first binomial by the third and this result by the second binomial.

$$25. (n - 3)(n^2 + 9)(n + 3).$$

$$26. (t + 1)(t^2 + 1)(t^4 + 1)(t - 1).$$

$$27. (n - 2)(n^2 + 4)(n^4 + 16)(n + 2).$$

$$28. (m + 2)(m - 2)(m^2 - 4). \text{ ✖ }$$

$$29. (a - bc)(a + bc)(a^2 - b^2c^2).$$

$$30. (x + 2)(x^2 + 4)(x - 2)(x^4 + 16).$$

$$31. (x^2 + y^2)(x - y)(x + y)(x^4 - y^4).$$

$$32. [n + r + t][n + r - t].$$

$$33. [n + r - 5][n + r + 5].$$

$$34. (2n + 3r)(n + r). \quad 39. (4k - 6nt)(7k + 2nt).$$

$$35. (2n - 3r)(2n + r). \quad 40. (n^2 - 3t)(4n^2 + 5t).$$

$$36. (3r - 5n)(2r - n). \quad 41. (4t^2 + h^2)(4t^2 - h^2).$$

$$37. (3rn - 2t)(4rn + t). \quad 42. (4t^2 - 2n^3)(4t^2 + 3n^3).$$

$$38. (5r^2n + h)(2r^2n - 3h). \quad 43. (n^2 + 2nt + t^2) \div (t + n).$$

$$44. (n^2 - 20nt + 100t^2) \div (10t - n).$$

$$45. (n^8 - 81) \div (n^4 + 9).$$

$$46. (t^2 + 3t + 2) \div (t + 2).$$

$$47. (n^2 - 5n + 6) \div (n - 3).$$

Indicate the simplest way to obtain the products of the following:

$$48. (n + t)(n + t)(n - t)(n - t).$$

$$49. (n^2 + r^2)(n + r)(n^4 + r^4)(n - r).$$

$$50. (4t^2 + n^2)(2t + n)(16t^4 + n^4)(2t - n).$$

CHAPTER XIV

FACTORING

69. Definitions. *Factoring* is the process of finding two or more expressions whose product is equal to a given expression.

Many simple exercises in factoring were solved in the preceding chapter in connection with the rules of multiplication there given. In fact, the process of factoring is the reverse of multiplication.

The subject of factoring is extensive. In this chapter we shall consider only the more common forms of factorable expressions, and usually seek only such factors as have integers as coefficients.

If a polynomial cannot be expressed as the product of expressions other than itself and 1, it is said to be *prime*.

Thus $2x + 1$, $3a - 5$, and $x^2 + 1$ are prime.

70. Square root of monomials. In factoring it is often necessary to find the square root, the cube root, and other roots of monomials.

The *square root* of a monomial is one of the two equal factors whose product is the monomial.

Since $+2 \cdot +2 = 4$ and $-2 \cdot -2 = 4$, the square root of 4 is ± 2 , which means both plus 2 and minus 2.

Similarly, the square root of 9 is ± 3 and the square root of a^2 is $\pm a$.

That is, *Every positive number or algebraic expression has two square roots which have the same absolute value but opposite signs.*

It is customary to speak of the positive square root of a number as the *principal square root*, and if no sign precedes the radical, the principal root is understood.

Thus, $\sqrt{4} = 2$, not -2 ; $-\sqrt{4} = -2$, not $+2$.

When both the positive and the negative square roots are considered, the double sign must precede the radical.

Since $x^3 \cdot x^3 = (-x^3)(-x^3) = x^6$, then $\pm \sqrt{x^6} = \pm x^3$.

That is, *The exponent of any letter in the square root of a monomial is one half the exponent of that letter in the monomial.*

Hence for extracting the square root of a monomial where both positive and negative factors are desired we have the

RULE. *Write the square root of the numeric coefficient preceded by the double sign $\pm$ and followed by all the letters of the monomial, giving to each letter an exponent equal to one half its exponent in the monomial.*

A rule similar to the preceding one holds for the fourth root, sixth root, and other even roots.

Thus, $\pm \sqrt[4]{81 c^8} = \pm 3 c^2$, and $\pm \sqrt[6]{a^{18}} = \pm a^3$.

In this chapter, and also in Chapter XVI on Fractions, where square roots arise, only the *principal* square root will be considered.

According to the definition of square root, the two factors of a term either of which is its square root *must be equal*. Consequently they must have the same sign.

ORAL EXERCISES

Perform the indicated operation in the following:

- | | | |
|-----------------------------|---------------------------------|--|
| 1. $\sqrt{2^2}$. | 11. $\sqrt{4 x^2 a^4}$. | 21. $\sqrt{225 a^4 x^8}$. |
| 2. $\sqrt{5^2}$. | 12. $\sqrt{36 a^2 x^6}$. | 22. $\sqrt{2^2 \cdot 3^2 \cdot 7^2}$. |
| 3. $\sqrt{3^4}$. | 13. $\sqrt{64 a^2 x^4}$. | 23. $\sqrt{9 \cdot 25 \cdot 81}$. |
| 4. $\sqrt{2^2 \cdot 3^2}$. | 14. $\sqrt{49 a^4 x^6}$. | 24. $\sqrt{3^2 \cdot 16 \cdot 5^2}$. |
| 5. $\sqrt{4^2 \cdot 7^2}$. | 15. $\sqrt{81 a^8 x^2}$. | 25. $\sqrt{4 x^2 (a^2)^2}$. |
| 6. $\sqrt{4 a^2}$. | 16. $\sqrt{169 a^6 x^2}$. | 26. $3 \sqrt{x^4 (y^3)^2}$. |
| 7. $\sqrt{9 x^4}$. | 17. $\sqrt{144 a^2 x^8}$. | 27. $ax \sqrt{(x+a)^2}$. |
| 8. $\sqrt{16 x^2}$. | 18. $6 \sqrt{100 a^{10}}$. | 28. $3 \sqrt{4(x-a)^2}$. |
| 9. $\sqrt{25 x^6}$. | 19. $5 \sqrt{121 a^{20}}$. | 29. $\sqrt{9 x^2 (x+a)^2}$. |
| 10. $\sqrt{9 a^2 x^2}$. | 20. $2 \sqrt{196 a^{12} x^4}$. | 30. $2 \sqrt{(3x-2)^2 a^4}$. |

71. Polynomials with a common monomial factor. The type form is

$$ab + ac - ad.$$

Since $ab + ac - ad = a(b + c - d)$,

we have, for factoring expressions having a common monomial factor, the following

RULE. Determine by inspection the monomial factor which is the product of all numeric and literal factors common to every term of the polynomial.

Divide the polynomial by this monomial factor.

Write the quotient in a parenthesis preceded by the monomial factor.

EXAMPLE

Factor $6 a^3 x - 15 a^2$.

Solution. The common monomial factor of both terms is seen to be $3 a^2$. Dividing the binomial by $3 a^2$ the quotient is $2 ax - 5$.

Therefore $6 a^3 x - 15 a^2 = 3 a^2 (2 ax - 5)$.

$$1-2a+b=2(a+b)$$

FACTORING

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ORAL EXERCISES

Factor the following :

- | | | |
|------------------|------------------------|--------------------------------|
| 1. $2x - 2.$ | 11. $ax - cx.$ | 21. $2ac + abc.$ |
| 2. $2x + 4.$ | 12. $ac - c^2.$ | 22. $3ax - 6acx.$ |
| 3. $3x - 6.$ | 13. $3ax - 6a.$ | 23. $4ac - 12a.$ |
| 4. $5x - 10.$ | 14. $4a^2 - 12a.$ | 24. $ax - bx^2.$ |
| 5. $ax + a.$ | 15. $2\pi R - 2\pi r.$ | 25. $a^2x + ax.$ |
| 6. $3ax - 6a.$ | 16. $ax + acx.$ | 26. $2ax - 4ab.$ |
| 7. $7a - 21.$ | 17. $3ax - 6ax^2.$ | 27. $3ax + 6bx.$ |
| 8. $9 - 3a.$ | 18. $5ax^2 - 10x^2.$ | 28. $2ax^2 + a^2x = x(2x + a)$ |
| 9. $5x^2 + 10x.$ | 19. $12ax - 10bx.$ | 29. $acx - 3ac + c.$ |
| 10. $cx - 2c.$ | 20. $2a^2x + ax^2.$ | 30. $am + an + ax.$ |

EXERCISES

Write the prime factors of :

- | | | |
|--|-----------------------------------|--------------------------------------|
| 1. $2a + 6.$ | 6. $2\pi r - 2\pi a.$ | 11. $ab - ax - ac.$ |
| 2. $ax + x^2.$ | 7. $a^2b - ab^2.$ | 12. $\pi R^2h + \pi r^2h + \pi Rrh.$ |
| 3. $a^3 + a^2.$ | 8. $2\pi Rl + 2\pi rl.$ | 13. $2a - 4a^2 + 6a^3.$ |
| 4. $3ax - 15a^2.$ | 9. $3ax - 6ay.$ | 14. $a^5 + a^2 - a^4.$ |
| 5. $2a^3 + 6a.$ | 10. $4a^2 + 16ab.$ | 15. $3c^2 - 12c - 18c^4.$ |
| 16. $ax - a^2x^2 - a^3x.$ | 19. $3c^2 - 15c + 6c^3.$ | |
| 17. $ay - abc - aby.$ | 20. $a^4 - a^3 + a^2 + a.$ | |
| 18. $a^2 - 2ax + a.$ | 21. $8a^2 - 4a^6 + 12a^3 - 6a^5.$ | |
| 22. Solve for x , $ax = a(b + c).$ | | |
| 23. Solve for x , $ax = am - ac.$ | | |
| 24. Solve for y , $my = ma + bm + cm.$ | | |
| 25. Solve for y , $ay = ab - abc - a^2.$ | | |
| 26. Solve for a , $ax = bx - cx + x^2.$ | | |
| 27. Solve for z , $az = a^2 - abc - 2a.$ | | |

Factor, then compute the numeric value of:

28. $3 \cdot 5 \cdot 17^2 - 3 \cdot 5 \cdot 4^3$.

HINT. $3 \cdot 5 \cdot 17^2 - 3 \cdot 5 \cdot 4^3 = 3 \cdot 5(17^2 - 4^3)$, etc.

29. $(3.1416)196 - (3.1416)121$.

30. $7 \cdot 13(31)^2 + 5 \cdot 13(31)^2 - 3 \cdot 13(31)^2$.

31. $\frac{22}{7}(21)^2 + \frac{22}{7}(14)^2 - \frac{22}{7}(21 \cdot 14)$.

32. $3.1416 \cdot (13)^2 + 3.1416(65) + \frac{4(3.1416)(13)(11)}{2}$.

72. ^{4-b terms}Polynomials having a common binomial factor. The type form is

$$ax + ay + bx + by.$$

Plainly, $ax + ay + bx + by = a(x + y) + b(x + y)$.

Dividing both terms of $a(x + y) + b(x + y)$ by $(x + y)$, the quotient is $a + b$.

Therefore $ax + ay + bx + by = (x + y)(a + b)$.

EXAMPLE

Factor $2ac + 5bx + bc + 10ax$.

Solution. $2ac + bc + 5bx + 10ax = c(2a + b) + 5x(b + 2a)$
 $= (2a + b)(c + 5x)$.

The preceding example illustrates the

RULE. *Arrange the terms of the polynomial to be factored in groups of two or more terms each such that in each group a monomial factor may be written outside a parenthesis, which in each case will contain the same expression.*

Rewrite, placing these monomial factors outside parentheses.

Then divide by the expression in parenthesis and write the divisor as one factor and the quotient as the other.

Polynomials which may be factored by grouping terms according to the foregoing rule usually contain either four, six, or eight terms.

It is important to note that one can obtain two apparently different sets of factors for a given expression. Thus

$$(a - b)(x - 3y) = (b - a)(3y - x),$$

for each pair by actual multiplication gives

$$ax - bx - 3ay + 3by.$$

An inspection of the expression shows that the two binomials of the first pair are the negatives respectively of those in the second pair; hence either pair of factors is correct.

The relation that the process of factoring bears to the processes of multiplication and division of monomials and polynomials should be kept constantly in mind. In multiplication we have two factors given and are required to find their product. In division we have the product and one factor given and are required to find the other factor. In factoring, however, the problem is a little more difficult, for we have only the product given, and the insight which can be obtained only by experience will enable us to determine the factors. To gain this insight rapidly one must keep clearly in mind the typical products of Chapter XIII and must study carefully certain typical forms, two of which have already been presented on pages 146 and 148.

There is no simple operation the performance of which makes us sure that we have found the *prime* factors of a given expression. Here, again, only the ability which comes with practice and experience enables us to find prime factors with certainty.

A partial check, however, that may be applied to all the exercises in factoring consists in actually multiplying together the factors that have been found. The result should be the original expression.

ORAL EXERCISES

1. Divide $3(a - b) + c(a - b)$ by $a - b$. What is the quotient? What are the factors of the first expression?

Give the binomial factors for :

- | | |
|----------------------------|------------------------------|
| 2. $3(a - b) + c(a - b)$. | 4. $x(a - 2b) + y(a - 2b)$. |
| 3. $a(x + 2) + b(x + 2)$. | 5. $a(m - 2n) - b(m - 2n)$. |

Factor :

- | | |
|--|-------------------------------|
| 6. $x(a - b) + y(a - b)$. | 9. $(s + t)x + (t + s)y$. |
| 7. $2n(a - 7) + (a - 7)$. | 10. $a(n - 7) - b(-7 + n)$. |
| 8. $(c - a)a - (c - a)c$. | 11. $x(r - t) + 2y(-t + r)$. |
| 12. $b(a - x) - (a - x)$. | |
| 13. $2m(a - 2t) + (a - 2t)$. | |
| 14. $3m(a - 2c) - n(a - 2c)$. | |
| 15. $(x - y)m + n(x - y)$. | |
| 16. $r(t - x) - (t - x)m$. | |
| 17. $2a(x - m) + (x - m)5b$. | |
| 18. $4a(2x - 7) - 3(2x - 7)$. | |
| 19. $2a(a - 3c) + (a - 3c)x$. | |
| 20. $7x(a - b) - 3(a - b)$. | |
| 21. $a(x - y) + b(x - y) - c(x - y)$. | |
| 22. $x(a - 3) + c(a - 3) + (a - 3)$. | |

EXERCISES

Factor the following:

1. $2m + bm + 2z + bz$. 8. $a^3 + 1 + a + a^2$.

HINT. Collecting the coefficients of m and z we have

9. $4x(a - c) + 3(c - a)$.

$(2 + b)m + (2 + b)z$, etc.

HINT. This can be written

$4x(a - c) - 3(a - c)$, etc.

2. $ax + 2x + ay + 2y$.

3. $rx + ry + sx + sy$.

10. $t(c - t) - x(t - c)$.

4. $ax + ay + 2x + 2y$.

11. $rx + x + ar + a$.

5. $ay + az + cy + cz$.

12. $ac + ax + x + c$.

6. $ah + bh + ak + bk$.

13. $28ac + 9rx + 21ar + 12cx$.

7. $x^3 + x^2 + x + 1$.

14. $2ax + 7a^2 + 16bx + 56ab$.

15. $36by + 45bd + 4y + 5d$.

16. $5x^3 + 10x + 5x^2 + 10$.

17. $3b + 5by^3 + 6hy + 10hy^4$.

18. $a(r - t) - c(t - r)$.

19. $2x(c - t) + 3(t - c)$.

20. $3ax - 6bx - ay + 2by$.

Solution. $3ax - 6bx - ay + 2by$

$3x(a - 2b) - y(a - 2b) = (a - 2b)(3x - y)$.

21. $ax + 3a - bx - 3b$.

22. $6ax - 2ac + 3x - c$.

23. $ac - ax + bx - bc$.

24. $2an + 9bm - 3am - 6bn$.

25. $m^2 - at + amt - m$. 29. $r^3 + at^2 + r^2at + rt$.

26. $6rx - 4rc - 6c + 9x$. 30. $7he - 77hz - de + 11dz$.

27. $a^3 - 2ax - a^2x + 2x^2$. 31. $12ax^5 - 6ax^3 - 4x^5 + 2x^3$.

28. $a^2c - acx + acx^2 - a^2cx$. 32. $4dr - 4rs + dst - d^2t$.

$$33. 16 acxy - 40 bx^2y - 6 abc^2 + 15 b^2cx.$$

$$34. 36 axy + 45 acy - 4 xy - 5 cy.$$

$$35. \text{Solve for } x, x(a+1) = b(a+1) + c(a+1).$$

$$36. \text{Solve for } x, ax + bx = ac + bc + 2a + 2b.$$

HINT. In the preceding equation, and in similar ones, the value of the unknown should be obtained by the use of factoring. In the final step of the solution mental division should be employed, not ordinary long division.

$$37. \text{Solve for } z, abz + z = abr + r + 2ab + 2.$$

$$38. \text{Solve for } y, ay - 3y = am - 3m - 6 + 2a.$$

$$39. \text{Solve for } x, 2ax + cx = 2ab + cm + bc + 2am.$$

$$40. \text{Solve for } x, ax + bc = a^2 + ab - ac - bx.$$

$$41. \text{Solve for } x, ax + 2c = ac + 3a - 6 + 2x.$$

73. Trinomials which are perfect squares. Here the type form is

$$a^2 \pm 2ab + b^2.$$

This, as on page 134, gives us the two expressions:

$$a^2 + 2ab + b^2 = (a+b)^2, \quad a^2 - 2ab + b^2 = (a-b)^2.$$

If an algebraic expression is the product of two equal factors, it is said to be a *perfect square*.

A *trinomial*, arranged according to the descending powers of one letter, is a *perfect square* if the first and third terms are positive and if the absolute value of the middle term is twice the product of the absolute values of the square roots of the other two terms.

Thus, in the type form above, the middle term $2ab = 2 \cdot \sqrt{a^2} \cdot \sqrt{b^2}$.

Similarly, the trinomial $25a^4 + 60a^2b + 36b^2$ is a perfect square, since the middle term $60a^2b = 2 \cdot \sqrt{25a^4} \cdot \sqrt{36b^2} = 2 \cdot 5a^2 \cdot 6b$.

ORAL EXERCISES

Form perfect trinomial squares by supplying the missing term in :

1. $a^2 + (?) + 1$.
2. $a^2 + (?) + m^2$.
3. $a^2 + (?) + 4$.
4. $x^2 + (?) + 25$.
5. $x^2 + (?) + 9t^2$.
6. $1 + (?) + 16c^2$.
7. $x^2 + 2cx + (?)$.
8. $x^2 + 4ax + (?)$.
9. $x^2 + 10ax + (?)$.
10. $a^2 - 6ax + (?)$.
11. $n^2 - 18n + (?)$.
12. $a^2 - 14ax + (?)$.
13. $c^2 + 8c + (?)$.
14. $c^2 - 5ac + (?)$.
15. $x^4 + 12x^2c + (?)$.
16. $x^4 - 12x^2 + (?)$.
17. $4x^2 + 4x + (?)$.
18. $9x^2 - 6x + (?)$.
19. $4x^2 - 20cx + (?)$.
20. $16a^2 - 16ax + (?)$.
21. $9x^2 - 18ax + (?)$.
22. $9x^2 + 24cx + (?)$.
23. $16x^2 - 32mx + (?)$.
24. $(?) + 12x + 36$.
25. $(?) - 16x + 64$.
26. $(?) - 24ac + 9c^2$.
27. $16x^2 - (?) + 4y^2$.
28. $25a^2 + 10a + (?)$.

For obtaining one of the two equal factors of a perfect trinomial square we have the

RULE. *Arrange the terms of the trinomial according to the descending powers of some letter in it.*

Extract the square roots of the first and third terms and connect the results by the sign of the middle term.

Before applying the foregoing rule one should never fail to observe whether the expression to be factored is a perfect trinomial square or not.

ORAL EXERCISES

Factor the following :

1. $x^2 - 4x + 4$.
2. $x^2 + 6x + 9$.
3. $x^2 - 10x + 25$.
4. $a^2 - 12a + 36$.
5. $4a^2 - 4a + 1$.
6. $9a^2 + 6a + 1$.
7. $9c^2 - 12c + 4$.
8. $a^2 - 14a + 49$.
9. $9x^4 - 18x^2 + 9$.

- | | |
|--------------------------|---------------------------------|
| 10. $c^2 + 4 - 4c$. | 19. $c^8 + 2c^4 + 1$. |
| 11. $x^2 + 9 - 6x$. | 20. $9t^2 - 42t + 49$. |
| 12. $36t^2 - 12t + 1$. | 21. $25x^2 - 20x + 4$. |
| 13. $64 - 16m^2 + m^4$. | 22. $9t^2 - 30t + 25$. |
| 14. $81 - 18t^3 + t^6$. | 23. $121t^2 - 44t + 4$. |
| 15. $100 - 20t + t^2$. | 24. $t^{10} - .4t^5 + .04$. |
| 16. $1 + 6t^2 + 9t^4$. | 25. $x^2 + x + \frac{1}{4}$. |
| 17. $a^6 + 4a^3 + 4$. | 26. $4x^2 + x + \frac{1}{16}$. |
| 18. $t^4 - 10t^2 + 25$. | 27. $9a^2 - 2a + \frac{1}{9}$. |

EXERCISES

Factor the following:

- | | |
|--|---|
| 1. $x^2 + 2xy + y^2$. | 12. $144t^2 - 120at + 25a^2$. |
| 2. $a^2 - 2at + t^2$. | 13. $169x^2 + 78ax + 9a^2$. |
| 3. $a^2 - 10at + 25t^2$. | 14. $36m^2 + 25n^2 - 60mn$. |
| 4. $16t^2 - 8at + a^2$. | 15. $x^4 + 4y^4 + 4x^2y^2$. |
| 5. $4x^2 - 12xy + 9y^2$. | 16. $4a^2t^8 - 4at^4 + 1$. |
| 6. $25x^2 - 20xy + 4y^2$. | 17. $12a^4t^3 + 4a^8 + 9t^6$. |
| 7. $9a^2 - 30at + 25t^2$. | 18. $121x^2 - 110cx + 25c^2$. |
| 8. $4a^2 - 28at + 49t^2$. | 19. $9x^4 - 66x^2 + 121$. |
| 9. $9t^2 - 12at + 4a^2$. | 20. $\frac{25}{16}x^2 + \frac{5}{2}x + 1$. |
| 10. $x^2 + \frac{2}{3}x + \frac{1}{9}$. | 21. $y^2 + y + \frac{1}{4}$. |
| 11. $a^2 - 22ax + 121x^2$. | 22. $36a^2 + 12a + 1$. |
| 23. $121x^2z^2 - 220txz + 100t^2$. | |
| 24. $169a^2 - 156az + 36z^2$. | |

Solve for x :

25. $(a + b)x = a^2 + 2ab + b^2$.
 26. $ax - cx = a^2 - 2ac + c^2$.
 27. $ax - 2tx = a^2 - 4at + 4t^2$.

Solve for z :

$$28. nz - 4z = n^2 - 8n + 16.$$

$$29. az - c^2 = a^2 + 2ac - cz.$$

$$30. az + 2an = a^2 + nz + n^2.$$

It is only in the beginning of factoring that polynomials are classified for the student. In the practical work of handling fractions and solving equations he must determine for himself the type of the polynomial to be factored. It is therefore very important that he fix in mind the various types and the manner of factoring each. Moreover, he should remember that the polynomials which arise in practice often have three or more factors. Miscellaneous review exercises afford excellent practice in recognizing types, and in determining *all* the prime factors.

At this point the suggestions given on page 167 will prove helpful, though only the first two of the types there given have as yet been considered.

REVIEW EXERCISES

Separate into prime factors:

$$(1. 3x^2 + 18x + 27.$$

$$2. a^3 + 2a^2 + a.$$

$$3. 2t^4 - 12t^2 + 18.$$

$$4. t^5 - 4t^4 + 4t^3.$$

$$5. a^2x^2 - 2a^2x + a^2.$$

$$6. 12t^2 + 36t + 27.$$

$$7. 2c^3 - 20c^2 + 50c.$$

$$8. 4x^2 - 16c^2x + 16c^4.$$

$$9. 8a^3 - 40a^2t + 50at^2.$$

$$10. 5at^4 - 70at^2 + 245a.$$

$$11. 32at^3 - 48at^6 + 18at^4.$$

$$12. 7t^4 + 28t^2 - 28t^3.$$

$$13. 14ant - 7n - 21bnz.$$

$$14. 2at + 2ax + 2bt + 2bx.$$

$$15. 3ct - 3cx + 3at - 3ax.$$

$$16. at - a + t - 1.)$$

$$17. x^4 + x^3 + x^2 + x.$$

$$18. 2a^5 - 2a^4 + 2a^3 - 2a^2.$$

$$19. 30at - 15a + 17c - 34ct.$$

$$20. 6atx + 6btx - 10aty - 10bty.$$

$$21. 56x^2 + 63xc - 40xt - 45ct.$$

Solve for x :

$$22. (a + b)x = c(a + b) - d(a + b).$$

$$23. ax - bx = a^2 - 2ab + b^2.$$

$$24. ax - cx = ab + an - cn - bc.$$

$$25. ax + 3x = ab - 6t + 3b - 2at.$$

$$26. tx - 4c^2 = t^2 - 4ct + 2cx.$$

$$27. at + t + mx = am + m + tx.$$

$$28. 2bct = bt^2 + bc^2 - bcx + btx.)$$

Factor:

$$29. \pi R^2 - \pi r^2 + 4\pi(R + r)^2.$$

$$30. [\pi R^2 + \pi(R + r)^2 - \pi r^2] 2h.$$

$$31. (\pi R^2 + \pi r^2) 2h - (\pi R^2 + \pi r^2) h.$$

$$32. \pi h(R - r)^2 - \pi h(R + r)^2 + \pi h R^2.$$

74. A binomial the difference of two squares. The type form is

$$a^2 - b^2.$$

By page 137, $a^2 - b^2 = (a + b)(a - b).$

Hence we have the

RULE. Extract the square root of the absolute value of each term of the binomial.

Add the two square roots for one factor, and subtract the second from the first for the other.

ORAL EXERCISES

Factor:

- | | | |
|---|---------------------------------|-------------------------------|
| 1. $a^2 - x^2$. | 9. $9t^2 - a^2$. | 17. $144t^2 - 49r^2$. |
| 2. $x^2 - 1$. | 10. $\pi R^2 - \pi r^2$. | 18. $64x^2 - 169t^2$. |
| 3. $x^2 - 4$. | 11. $9t^4 - 4x^2$. | 19. $81r^2 - 121m^4$. |
| 4. $4x^2 - 1$. | 12. $25t^2 - 36m^2$. | 20. $.81t^2 - 1.96r^2$. |
| 5. $a^2 - 9$. | 13. $49t^2 - 64r^2$. | 21. $169r^4 - .36t^2$. |
| 6. $1 - m^2$. | 14. $100x^4 - 81t^2$. | 22. $\frac{1}{25} - .04x^2$. |
| 7. $16 - t^2$. | 15. $100t^2 - 9x^2$. | 23. $.25 - \frac{1}{4}a^2$. |
| 8. $4t^2 - 9$. | 16. $4\pi R^2h - \pi r^2h$. | 24. $m^2 - \frac{25}{64}$. |
| 25. $2.25a^2 - 1.44b^2$. | 28. $x^2 - \frac{1}{9}$. | |
| 26. $\frac{49}{36}x^2 - \frac{16}{25}y^2$. | 29. $\frac{4}{9}x^2 - 1.21$. | |
| 27. $.36a^2b^2 - .25c^2$. | 30. $\frac{4}{9}y^2 - .16z^2$. | |

Solve for x :

- | | |
|------------------------------|------------------------------------|
| 31. $x(t - 2) = t^2 - 4$. | 34. $(m - 6)x = m^2 - 36$. |
| 32. $x(3c + 1) = 9c^2 - 1$. | 35. $(2t - 3)x = 4t^2 - 9$. |
| 33. $x(n - a) = n^2 - a^2$. | 36. $(4r - 5n)x = 16r^2 - 25n^2$. |

EXAMPLES

1. Factor: $x^4 - 81$.

Solution. $x^4 - 81 = (x^2 + 9)(x^2 - 9)$, by application of the rule
 $= (x^2 + 9)(x + 3)(x - 3)$, by application of
the rule to $x^2 - 9$.

2. Factor: $16x^8 - 81a^{12}$.

Solution. $16x^8 - 81a^{12} = (4x^4 + 9a^6)(4x^4 - 9a^6)$
 $= (4x^4 + 9a^6)(2x^2 + 3a^3)(2x^2 - 3a^3)$.

EXERCISES

Factor:

- | | | |
|------------------|---------------------|----------------------|
| 1. $a^4 - x^4$. | 6. $15 - 16 m^4$. | 11. $x^4 - 625$. |
| 2. $a^4 - 1$. | 7. $16 - a^8$. | 12. $a^8 - b^4$. |
| 3. $t^4 - 16$. | 8. $a^{12} - 81$. | 13. $y^4 - a^8$. |
| 4. $x^4 - 81$. | 9. $t^4 - 256$. | 14. $a^4 - 16 b^4$. |
| 5. $81 - t^8$. | 10. $16 t^4 - 81$. | 15. $a^4 t^8 - 81$. |

16. $(a + x)^2 - t^2$.

HINT. $(a + x)^2 - t^2 = [(a + x) + t][(a + x) - t]$, etc.

- | | |
|----------------------------------|---------------------------------------|
| 17. $(a + t)^2 - 4$. | 31. $16 a^2 - 9(m + t)^2$. |
| 18. $(x - 3)^2 - m^2$. | 32. $25 x^2 - (a - 2 x)^2$. |
| 19. $(2 x + t)^2 - 9 a^2$. | 33. $4 r^2 t^2 - (2 r - t)^2$. |
| 20. $4(x - 2 a)^2 - 25$. | 34. $36 a^2 t^2 - (at - 2 x)^2$. |
| 21. $9(t + 3 x)^2 - 4 m^2$. | 35. $16 m^2 - 4 a^2(x - 1)^2$. |
| 22. $25 a^2(x - t)^2 - 16 m^2$. | 36. $81 t^2 - 16(2 t - 3 x)^2$. |
| (23. $a^2 - (2 x - t)^2$. | 37. $64 x^2 - 9(x - 3 t)^2$. |
| 24. $m^2 - (3 a - c)^2$. | 38. $(a + x)^2 - (b + y)^2$. |
| 25. $4 x^2 - (2 a + 3)^2$. | 39. $(a - x)^2 - (b - t)^2$. |
| 26. $16 m^2 - (5 a - x)^2$. | 40. $(a - 2 x)^2 - (3 x + t)^2$. |
| 27. $25 x^2 - (a + 7)^2$. | 41. $(2 a - t)^2 - (2 x - m)^2$. |
| 28. $a^2 x^2 - (a - x)^2$. | 42. $(3 t - x)^2 - (m + 3 a)^2$. |
| 29. $a^2 t^2 - (t + a)^2$. | 43. $(2 a - 3 b)^2 - (3 c - 2 x)^2$. |
| 30. $9 a^2 - 4(t - x)^2$. | 44. $(3 x - 1)^2 - (1 - 2 y)^2$. |

Factor, then find the numeric value of:

45. $(12.7)^2 - (9.8)^2$.

Solution. $(12.7)^2 - (9.8)^2 = (12.7 + 9.8)(12.7 - 9.8)$
 $= (22.5)(2.9) = 65.25.$

46. $(7.3)^2 - (4.3)^2$.

50. $(9.4)^2 - (6.3)^2$.

47. $(8.6)^2 - (5.6)^2$.

51. $(14.1)^2 - (7.8)^2$.

48. $(11.4)^2 - (6.4)^2$.

52. $7^2 - 2 \cdot 7 \cdot 5 + 5^2$.

49. $(13.7)^2 - (3.7)^2$.

53. $49 - 2 \cdot 7 \cdot 9 + 81$.

54. If $\pi = 3.1416$, $r = 7$, and $R = 11$, find the value of $\pi R^2 - \pi r^2$.

55. If $\pi = \frac{22}{7}$, $R = 4$, and $r = 3$, find the value of $\frac{4}{3} \pi R^3 - \frac{4}{3} \pi r^3$.

Some polynomials of four or six terms may be arranged as the difference of two squares and factored as in the preceding exercises.

EXERCISES

Factor:

1. $a^2 - 2ab + b^2 - x^2$.

Solution. $a^2 - 2ab + b^2 - x^2 = (a - b)^2 - x^2$

$= (a - b + x)(a - b - x)$.

2. $a^2 + 2at + t^2 - x^2$.

9. $6xy - t^2 + y^2 + 9x^2$.

3. $a^2 + 2ac + c^2 - 4m^2$.

10. $4x^2 - 25y^2 + 9t^2 - 12tx$.

4. $a^2 - 4ax + 4x^2 - 9t^2$.

11. $x^2 - t^4 + 16m^2 - 8mx$.

5. $4x^2 - 4x + 1 - m^2$.

12. $1 - 4at - t^2 + 4a^2t^2$.

6. $25 - 10t - 4m^2 + t^2$.

13. $12ab - 4m^2 + 9a^2 + 4b^2$.

7. $x^2 - 16n^2 - 4tx + 4t^2$.

14. $9m^2 - 25n^2 - 6mt + t^2$.

8. $y^2 - t^2 + x^2 - 2xy$.

15. $12at - 4a^2 - 9t^2 + m^2$.

Solution. $12at - 4a^2 - 9t^2 + m^2 = m^2 - (4a^2 - 12at + 9t^2)$

$= m^2 - (2a - 3t)^2$

$= [m + (2a - 3t)][m - (2a - 3t)]$

$= (m + 2a - 3t)(m - 2a + 3t)$.

16. $m^2 - a^2 - 2at - t^2$. 20. $4x^2 - a^2 + 4at - 4t^2$.
 17. $n^2 - b^2 - 2bd - d^2$. 21. $6m + 9x^2 - 9 - m^2$.
 18. $t^2 - b^2 + 4bc - 4c^2$. 22. $4ab - 4a^2 + 4m^6 - b^2$.
 19. $2tx - t^2 - x^2 + a^4$. 23. $a^2 + 2ab + b^2 - x^2 - 2tx - t^2$.
 24. $x^2 - 4xt + 4t^2 - 9a^2 + 6a - 1$.
 25. $1 + 2bc + 2a - c^2 - b^2 + a^2$.
 26. $a^2 - 1 + t^2 - m^2 + 2at + 2m$.
 27. $4a^2 + 10x - 25 - 12am - x^2 + 9m^2$.

REVIEW EXERCISES

Factor:

1. $a^3 - a$. 11. $tn^5 - t^5n$.
 2. $t^4 - 2t^2 + 1$. 12. $36a - 12a^3 + a^5$.
 3. $x^4 - 8x^2 + 16$. 13. $\frac{1}{2}atm + \frac{1}{2}at$.
 4. $a - a^5$. 14. $t^3 - t^2 - 4t + 4$.
 5. $t^8 - 2t^4 + 1$. 15. $3at + 3t^2 - 27t - 27a$.
 6. $a^5 - 18a^3 + 81a$. 16. $2a^3x + 3a^2x - 8ax - 12x$.
 7. $a^3 - a + a^2x - x$. 17. $na^2 - nt^2 + 4nt - 4n$.
 8. $2a^3 - 18at^2$. 18. $a^2x - 9x + a - 3$.
 9. $t^4 - 10t^2 + 9$. 19. $2x^4 - 40x^2 + 200 - 18$.
 10. $t^3 + 2t^5 + t^7$. 20. $5a^4 + 20a^3 + 5a^5 + 20a^2$.
 21. $a^2b^2 - b^2x^2 - 4a^2 + 4x^2$.
 22. $\frac{1}{4} + r + r^2 - s^2 - \frac{2st}{3} - \frac{t^2}{9}$.

Solve for x :

23. $ax = a^3b - ab$. 26. $ax - 4t^2 = a^2 - 4at + 2tx$.
 24. $ab = ax - 4a^2t$. 27. $t^2x - t^3 = tx - t$.
 25. $ax + bx = a^2 - b^2$. 28. $x(t - 3)(t^2 + 9) = t^4 - 81$.

FACTORING

$$(p^2 - \frac{1}{4}q^2) = (p + \frac{1}{2}q)(p - \frac{1}{2}q) \quad 161$$

75. The quadratic trinomial. The type form is

$$x^2 + bx + c.$$

For many trinomials of this type two binomial factors may be found of the form $(x + r)(x + s)$. The method of factoring to be used is the reverse of the method of multiplying two binomials given on page 139. From a study of the four examples there given it is evident that to factor a trinomial of the form $x^2 + bx + c$ we must, if possible, find two numbers whose product is c and whose sum is b . Let these numbers be called r and s .

Then the required factors are $x + r$ and $x + s$,
since $(x + r)(x + s) = x^2 + (r + s)x + rs$.

EXAMPLES

1. Factor $x^2 + 12x + 32$.

Solution. $x^2 + 12x + 32 = (x + ?)(x + ?)$.

It is necessary to find two numbers whose product is $+32$ and whose sum is $+12$.

Now $32 = 1 \cdot 32 = 2 \cdot 16 = 4 \cdot 8$.

The first two pairs are rejected, for each fails to give the sum $+12$. The third pair of factors of 32, namely 4 and 8, gives the correct sum.

Therefore $x^2 + 12x + 32 = (x + 4)(x + 8)$.

2. Factor $a^2 - 11a + 24$.

Solution. Since 24 is positive, its two factors must have the same sign; since -11 is negative, both factors must be negative. Now $24 = 1 \cdot 24 = 2 \cdot 12 = 3 \cdot 8 = 4 \cdot 6$. By inspection of these products, -3 and -8 are found to be the required numbers.

Therefore $a^2 - 11a + 24 = (a - 3)(a - 8)$.

3. Factor $c^2 - c - 42$.

Solution. The product -42 is negative; hence the required factors have *unlike* signs. Since -1 is negative, the negative factor of -42 must have the greater absolute value. Now $42 = 1 \cdot 42 = 2 \cdot 21 = 3 \cdot 14 = 6 \cdot 7$. We see that $6 + (-7) = -1$.

Therefore $c^2 - c - 42 = (c + 6)(c - 7)$.

EXERCISES

Factor:

- | | | |
|-----------------------|------------------------|-------------------------|
| 1. $x^2 + 3x + 2$. | 13. $a^2 - 9a + 14$. | (25. $t^2 + 3t - 10$. |
| 2. $x^2 + 4x + 3$. | 14. $n^2 + 6n + 9$. | 26. $t^2 - 2t - 15$. |
| 3. $x^2 + 5x + 4$. | 15. $n^2 + 7n + 12$. | 27. $t^2 + 2t - 15$. |
| 4. $x^2 + 6x + 5$. | 16. $h^2 + 8h + 15$. | 28. $t^2 - 4t - 21$. |
| 5. $x^2 + 7x + 6$. | 17. $x^2 - 9x + 20$. | 29. $t^2 + 4t - 21$. |
| 6. $x^2 - 8x + 7$. | 18. $t^2 - 45 + 4t$. | 30. $t^2 - 4t - 45$. |
| 7. $x^2 - 9x + 8$. | 19. $n^2 - 13n + 42$. | 31. $x^2 - 1.1x + .3$. |
| 8. $a^2 + 4a + 4$. | 20. $t^2 - t - 6$. | 32. $t^2 + .5t + .06$. |
| 9. $a^2 + 5a + 6$. | 21. $t^2 - t - 12$. | 33. $t^2 - .7t + .12$. |
| 10. $a^2 + 6a + 8$. | 22. $x^2 + x - 20$. | 34. $x^2 + .2x - .15$. |
| 11. $a^2 + 7a + 10$. | 23. $x^2 - x - 30$. | 35. $n^2 - 1n + .09$. |
| 12. $a^2 - 8a + 12$. | 24. $h^2 - 3h - 10$. | 36. $x^2 - 1.6x - .8$. |
| | 37. $-x^2 + x + 20$. | |

Solution. $-x^2 + x + 20 = -(x^2 - x - 20)$

$$= -1(x - 5)(x + 4) = (5 - x)(x + 4).$$

38. $-n^2 - n + 12$.

42. $t^2 + t^4 - 110$.

39. $9 - 8n - n^2$.

43. $a^2b^2 - 3aby - 28y^2$.

40. $-2x - x^2 + 63$.

44. $n^6 - 11n^3 + 18$.

41. $80 - 2n - n^2$.

45. $a^2b^2 + 10abx^3 - 24x^6$.

$$= (+)(-)$$

$$+ = (-)$$

46. $9tx^4y^6 + t^2x^8 - 22y^{12}$. 49. $(a-x)^2 - 5(a-x) - 14$.
 47. $33n^2 - a^2x^2 - 8anx$. 50. $n^2 + (a+b)n + ab$.
 48. $(a+x)^2 + 3(a+x) + 2$. 51. $n^2 + (2a+5)n + 10a$.
 52. $(a-2t)^2 - 8(a-2t) + 15$.
 53. $16 - 17(2t-x) + (2t-x)^2$.

REVIEW EXERCISES

Factor:

- (1. $x^3 - 4x$. 12. $56t + t^2 - t^3$.
 2. $4ax^2 - 4a$. 13. $4ax^2 - 16a^3$.
 3. $a^4 + 6a^3 + 9a^2$. 14. $162a - 2a^5$.
 4. $t^3 - 5t^2 + 6t$. 15. $4t^5 + 4t^4 + t^3$.
 5. $3n^2 - 9n - 30$. 16. $5an^4 - 20n^5$.
 6. $3at^3 - 3at^2 - 6at$. 17. $t^8 - 17t^4 + 16$.
 7. $x^4 - .5x^2 + .04$. 18. $81a - 18a^3 + a^5$.
 8. $nt^4 - 7nt^2 + 12n$. 19. $n^4 - 143n^2 - 144$.
 9. $at^4 - 7at^2 - 18a$. 20. $4a^5 - 92a^3 - 200a$.
 10. $a^4 - 13a^2 + 36$. (21. $5an^4 - 5an^3 + 5an^2 - 5an$.
 11. $3n^4 - 21n^3 - 54n^2$. 22. $4at^3 - 8at^2 - 24at + 48a$.
 23. $10abc - 30bc + 20ac - 60c$.
 24. $6at^2 + 3t^2c + 18atn + 9ctn$.
 25. $4a^3 - 4ay^2 - 8ayz - 4az^2$.
 26. Solve for x , $(a-3)x = a^2 - 5a + 6$.
 27. Solve for x , $4x + ax = a^2 - 16$.
 28. Solve for y , $ny + 42 = n^2 - n - 6y$.
 29. Solve for y , $ay + a = a^2 + 5y - 20$.
 30. Solve for z , $az + 3ac = a^2 + 2cz + 2c^2$.)

Factor :

$$31. \pi h(R + r)^2 + \pi h(R^2 - r^2).$$

$$32. 2 \pi h(R^2 - r^2) + 3 \pi h(R - r)^2.$$

$$33. 5 \pi(R^2 h - 4 h) + 2 \pi(R + 2)^2 h.$$

Simplify and factor :

$$34. (\pi R^2 - \pi r^2)h + (\pi R^2 + \pi r^2)3 h.$$

$$35. 5 \pi(R^2 - r^2)h - 2 h(\pi R^2 - \pi r^2).$$

$$36. [\pi R^2 + \pi r^2 + \pi(R + r)^2]h - [\pi R^2 + \pi r^2 + \pi Rr]h.$$

76. The general quadratic trinomial. The type form is

$$ax^2 + bx + c.$$

For many trinomials of this type two binomial factors of the form $(hx + k)(mx + n)$ may be found. The method of factoring such trinomials is illustrated in the following

EXAMPLES

1. Factor $3x^2 + 7x + 2$.

$$\text{Solution. } 3x^2 + 7x + 2$$

$$= (?x + ?)(?x + ?).$$

To obtain the correct factors we must supply such numbers for the interrogation points in (1) and in (2) as will give

$$\begin{array}{r} ?x + ? \\ \hline \end{array} \quad (1)$$

$$\begin{array}{r} ?x + ? \\ \hline \end{array} \quad (2)$$

$$\begin{array}{r} 3x^2 + ?x \\ + ?x + 2 \\ \hline 3x^2 + 7x + 2 \end{array}$$

$3x^2$ for the product of the first two terms of the binomials, $+2$ for the product of the last two terms of the binomials, and $+7x$ for the sum of the cross products.

Now

$$3x^2 = 3x \cdot x,$$

and

$$+2 = 1 \cdot 2.$$

The factors 1 and 2 may be substituted for the interrogation points in (1) and (2) in either of the following ways:

$$\begin{array}{r} 3x + 2 \\ x + 1 \end{array} \quad (\text{Incorrect})$$

$$\begin{array}{r} 3x + 1 \\ x + 2 \end{array} \quad (\text{Correct})$$

The first pair is rejected, for it fails to give a product having the required middle term, $+7x$. The second pair gives the correct product.

$$\text{Therefore } 3x^2 + 7x + 2 = (3x + 1)(x + 2).$$

2. Factor $3x^2 - 13x - 10$.

Solution. $3x^2 = 3x \cdot x$.

$$-10 = 1 \cdot -10 = -1 \cdot 10 = 2 \cdot -5 = -2 \cdot 5.$$

Test the following pairs of binomials:

$$\begin{array}{cccccccc} 3x + 1 & x + 1 & 3x - 1 & x - 1 & x + 2 & 3x - 2 & x - 2 & 3x + 2 \\ x - 10 & 3x - 10 & x + 10 & 3x + 10 & 3x - 5 & x + 5 & 3x + 5 & x - 5 \end{array}$$

Only the last pair gives the desired product.

$$\text{Therefore } 3x^2 - 13x - 10 = (x - 5)(3x + 2).$$

After a little practice it will usually be found unnecessary to write down all the pairs of binomials that do not produce the required product.

If none of the pairs gives the required product, the given trinomial is prime.

EXERCISES

Factor:

1. $2x^2 + 5x + 2$.
2. $3x^2 + 5x + 2$.
3. $2t^2 + 5t + 3$.
4. $2x^2 + 7x + 3$.
5. $3x^2 + 7x + 2$.
6. $3a^2 + 10a + 3$.
7. $2n^2 + 7n + 6$.
8. $3t^2 + 8t + 4$.
9. $2a^2 + 9a + 9$.
10. $3x^4 + 11x^2 + 6$.
11. $2n^2 - 3n - 2$.
12. $2t^2 - 5t - 3$.
13. $2a^2 - a - 3$.
14. $2x^2 - x - 6$.
15. $3a^2 + 2a - 1$.

16. $3t^2 - 5t - 2.$

17. $3n^2 - 8n - 3.$

18. $3a^2 + a - 2.$

19. $3t^2 + 4t - 4.$

20. $3n^4 + 7n^2 - 6.$

21. $3x^2 - 11x + 6.$

22. $3t^2 - 14t + 8.$

23. $3a^2 - 17a + 10.$

24. $6n^2 - n - 2.$

25. $6r^2 - 13r + 6.$

26. $10m^2 - 19m + 6.$

27. $15a^6 - 4a^3 - 4.$

28. $10n^2 - 39n + 14.$

29. $14 + 41t^2 + 15t^4.$

30. $6 + 35x^2 - 31x.$

31. $35a^2 + a - 6.$

32. $15t^2 - 24at + 9a^2.$

33. $11a^2b^2 - 80abx + 21x^2.$

34. $18a^2x^2 - 41anx - 10n^2.$

35. $35at^2 + 62abt - 33ab^2.$

36. $10an^2 - 59arn - 39ar^2.$

37. $6x^2 - 25x - 51.$

38. $12a^2t^2 - 53at + 30.$

39. $35x^6 - 102x^3 + 55.$

40. $26a^2n^2 + 81an - 35.$

41. $36x^2 - 9x - 10. \quad]$

77. General directions for factoring. Since no general method of factoring can be stated in a few simple rules, the process must be learned by means of such type forms and typical solutions as are given in the preceding pages. When these have been once thoroughly mastered, readiness in factoring expressions which are represented by them becomes a matter of experience. Generally a student finds it comparatively easy to factor a list of exercises classified under a particular type form, yet a list of miscellaneous exercises he finds difficult. This usually indicates inability to determine the type of an expression from its appearance. Until the student, by careful study of the type forms, has acquired the ability to do this, he will make little progress. There are many types that are not included in this book, which the student who continues the study of algebra will meet later.

The following suggestions will prove helpful in solving the types here considered :

I. First look for a common monomial factor, and if there is one (other than 1) separate the expression into its greatest monomial factor and the corresponding polynomial factor.

II. Then by the form of the polynomial factor determine with which of the following types it should be classed and use the methods of factoring applicable to that type.

1. $ax + ay + bx + by.$

3. $a^2 - b^2.$

2. $a^2 \pm 2ab + b^2.$

4. $x^2 + bx + c.$

5. $ax^2 + bx + c.$

III. Proceed again as in II with each polynomial factor obtained until the original expression has been separated into its prime factors.

MISCELLANEOUS EXERCISES

Factor :

1. $2n^3 - 2n.$

13. $n^8 - 2n^4 + 1.$

2. $3a^3 - 75a.$

14. $16x - 8x^3 + x^5.$

3. $ax^4 - 81a.$

15. $10a^2 - 10a - 60.$

4. $\pi ar^2 - 2\pi rh.$

16. $x^2 - .1x - .02.$

5. $\pi R^2 - 3\pi aR.$

17. $9 + a^4 - 10a^2.$

6. $ax^4 - 16a.$

18. $a^4 + 36 - 13a^2.$

7. $a^4 - x^{16}.$

19. $t^2 + 2n^2t^2 + n^4t^2.$

8. $2x^8 - 2.$

20. $\pi R^2 + \pi r^2 + 2\pi Rr.$

9. $3x^2 - 27y^2.$

21. $.02a^2 - .3a - 2.$

10. $n^2 - n^6.$

22. $x^3 - x^2 - 90x.$

11. $t^4 - 64t^8.$

23. $4x^2 - 2xy - 30y^2.$

12. $1 - 2n^2 + n^4.$

24. $4x^2 - 20tx + 25t^2.$

25. $14n^2 + 16n^4 + 3.$

40. $1 + n^4 + n - n^2.$

26. $3x^2 + 10x - 8.$

41. $x^5 - x^3 + x - 1.$

27. $2ab - a^2 - b^2.$

42. $t^5 + t - t^4 - t^3.$

28. $72x^2 - 96xy + 14y^2.$

43. $n^4 - (n^2 + 12an - 36a^2).$

29. $20 - x - x^2.$

44. $10a^2 + 5ab - 5b^2.$

30. $.5x^2 - .3x - .2.$

45. $x^2 - y^2 - x - y.$

31. $3a^2 + 5a - 28.$

46. $a - b + a^2 - b^2.$

32. $4n^4 - 9n^2 - 9.$

47. $n^4 - 30n^2 + 225 - 9m^2.$

33. $n^4 - 3n^3 + 4n^2 - 12n.$

48. $289 - 100x^2 - t^2 - 20tx.$

34. $2a^3t + 3a^2t - 8at - 12t.$

49. $x^3 + 3x^2 + 3x + 9.$

35. $.12a^2 - .7ab + b^2.$

50. $x^3 - 3x^2 + 9x - 27.$

36. $x^4 - x^2 - x + 1.$

51. $x^2 + 3.2xy + .6y^2.$

37. $n^5 - n^4 - n^3 + n^2.$

52. $n^3 + 3n^2t + 3nt^2 + 9t^3.$

38. $6a^4n - 3a^3bn - 3a^2b^2n.$

53. $18n^3 - 6n^2 + 3n - 1.$

39. $a^4 - a + 4 - 16a^2.$

54. $4ac - 2c + 8ad - 4d.$

55. Solve for x , $a^2 + 3x - ax - 9 = 0.$ $a + 9$

56. Solve for t , $a^2 + c^2 + ct = at + 2ac.$ $a - c$

57. Solve for z , $m^2 + 2m + 3z = mz + 15.$

58. Solve for x , $2a^2 + 3x = a + 2ax + 3.$

59. Solve for t , $a^2 - 3a - 2t = at - 2a + 6.$

CHAPTER XV

SOLUTION OF EQUATIONS BY FACTORING

78. Simple equations. A *simple* or *linear* equation in one unknown is one which may be put in such a form that

- (a) the unknown does not appear in any denominator;
- (b) only the first power of the unknown is involved.

Thus $5x - 2 = 13$, $3t - 7 = 5t - 21$, $ax + b = 0$, are simple equations. $(x + 4)(x - 5) = (x + 7)(x - 10)$ is also a simple equation, since on multiplying out it becomes $x^2 - x - 20 = x^2 - 3x - 70$, from which, after transposing, we get $2x + 50 = 0$.

In the preceding chapters only simple equations have been considered.

79. Quadratic equations. A *quadratic* equation in one unknown is one which may be put in such a form that

- (a) the unknown does not appear in any denominator;
- (b) the second but no higher power of the unknown is involved.

Thus $x^2 - 5x + 6 = 0$, $7x^2 + 4 = 6x + 16$, $ax^2 + bx + c = 0$ are quadratic equations.

A quadratic equation is often called an equation of the *second degree*.

The term in a quadratic equation which does not involve the unknown is called the *constant* term.

80. Solution of equations. The methods of factoring given in Chapter XIV enable us to solve many quadratic equations. In the solution of equations by factoring, use is made of the following

✓ **PRINCIPLE.** *If the product of two or more factors is zero, at least one of the factors must be equal to zero.*

Two or more, or even all, of the factors *may* be zero, but the vanishing of one is *sufficient* to make the product zero.

Consider the equation, in factored form,

$$(x - 3)(x - 8) = 0. \quad (1)$$

If this equation is to be solved, all the numbers which satisfy it must be found. That is, we must find every value of x for which the product on the left of (1) is zero. If the product is to equal zero, the foregoing principle requires that one of the factors be zero. Hence any value of x which satisfies (1) must make either $x - 3 = 0$ or $x - 8 = 0$. Hence x must equal either 3 or 8. On substituting 3 for x in (1) we obtain $(3 - 3)(3 - 8) = 0$, or $0 \cdot (-5) = 0$. Hence 3 is a root of (1). On substituting 8 for x in (1) we obtain $(8 - 3)(8 - 8) = 0$, or $5 \cdot 0 = 0$. Hence 8 is also a root of (1). A moment's inspection makes it clear that 3 and 8 are the only roots of the equation.

ORAL EXERCISES

For what value of x is each of the following expressions equal to zero?

1. $x - 3.$

4. $x + 9.$

7. $2x - 5.$

2. $x + 7.$

5. $2x - 10.$

8. $7x - 11.$

3. $x - 6.$

6. $3x - 18.$

9. $ax - b.$

10. What is the value of 3×0 ? of 0×3 ? of -7×0 ? of $n \times 0$? of $m \times 0$?

11. Make a statement which will include all the results of Exercise 10.

Find the value of:

12. $(x - 2)(x - 3)$ when $x = 0, 2, 4$.

13. $(x - 5)(x - 6)$ when $x = 1, 3, 5$.

14. $(x + 3)(x - 1)$ when $x = 1, 3, -3$.

15. $(2x - 1)(3x + 1)$ when $x = -\frac{1}{2}, \frac{1}{2}, \frac{1}{3}$.

16. $x(2x - 8)$ when $x = 3, 0, 4$.

In the following determine which of the numbers on the right is a root of the corresponding equation:

17. $(x - 2)(x - 1) = 0$. 1, 2, 3.

18. $(x - 3)(x - 4) = 0$. $-4, -3, 4, 3$.

19. $x(2x - 5) = 0$. $1, \frac{5}{2}, 0$.

20. $(2t - 3)(t + 1) = 0$. $0, 1, -1, \frac{3}{2}$.

21. $(t - 2)(t - 3)(t - 4) = 0$. 5, 2, 3, 4.

What are the roots of the following equations?

22. $(x - 2)(x + 2) = 0$.

23. $x(x - 3) = 0$.

24. $(x + 3)(x - 4) = 0$.

25. $(2t - 1)(3t - 2) = 0$.

26. $(3t + 1)(5t - 2) = 0$.

27. $(t - 1)(t - 3)(t - 4) = 0$.

28. $(2t - 1)(1 - t)(t + 2) = 0$.

29. Is there any one number which will make both factors of $(y - 5)(y + 7)$ equal to zero?

EXAMPLES

1. Solve $x^2 - 7x = 18$.

Solution. Transposing, $x^2 - 7x - 18 = 0$.

Factoring $(x - 9)(x + 2) = 0$.

The value of x which makes the first factor zero is a root of the quadratic. Setting $x - 9 = 0$, we obtain $x = 9$.

Similarly, that value of x which makes the second factor zero is also a root of the quadratic. Setting $x + 2 = 0$, we obtain $x = -2$.

Therefore $x = 9$ and $x = -2$ are the required roots.

Check. Substituting 9 for x in $x^2 - 7x = 18$ gives

$$81 - 63 = 18,$$

or $18 = 18$.

Substituting -2 for x in $x^2 - 7x = 18$ gives

$$4 + 14 = 18,$$

or $18 = 18$.

2. Solve $3x^2 = 5x$.

Solution. Transposing, $3x^2 - 5x = 0$.

Factoring, $x(3x - 5) = 0$.

Setting each factor equal to zero, we obtain $x = 0$ and $3x - 5 = 0$. Solving the second equation, $x = \frac{5}{3}$. Therefore the roots are $x = 0$ and $x = \frac{5}{3}$.

Check. $3 \times (0)^2 = 5 \times 0$, or $0 = 0$.

$$3\left(\frac{5}{3}\right)^2 = 5\left(\frac{5}{3}\right), \text{ or } \frac{25}{3} = \frac{25}{3}.$$

For solving an equation in one unknown by factoring we have the

RULE. *Transpose the terms so that the right member is zero. Then factor the expression on the left, set each factor which contains the unknown equal to zero, and solve the resulting equations.*

It must be kept in mind that a root of an equation is a number which satisfies the equation.

One should never divide each member of an equation by an expression containing the unknown, for if this is done one or more roots may be lost.

Thus, if in Example 2 we had divided both sides of the equation by x , the resulting equation would have been $3x - 5 = 0$, which, to be sure, gives us one root of the given equation. But we have lost the root $x = 0$, which corresponds to the factor by which we divided.

EXERCISES

Solve by factoring and check:

1. $x^2 = 4$.
2. $x^2 = 16$.
3. $x^2 = 49$.
4. $x^2 = 5x$.
5. $x^2 + 12 = 7x$.
6. $x^2 - x = 90$.
7. $x^2 + x = 56$.
8. $6x^2 - 18x = 0$.
9. $8 + x^2 = 9x$.
10. $5x^2 = 25x$.
11. $x^2 - 11x = 26$.
12. $x^2 = 19x - 34$.
13. $2x^2 + 5x + 3 = 0$.
14. $3x^2 + 5x + 2 = 0$.
15. $3x^2 = 11x + 20$.
16. $10x^2 = 13x + 9$.
17. $10x^2 = 19x - 6$.
18. $(x + 7)(x - 2) = 10$.
19. $(x + 3)(2x - 1) - 15 = 0$.
20. $(x - 3)^2 + (x + 3)(x - 3) = 0$.
21. $(2x - 1)^2 + (2x + 5)(2x - 1) = 0$.
22. $(3x - 2)^2 - (3x - 2)(x - 7) = 0$.
23. $x^2 - 4 = 3x + 6$.

Handwritten notes and symbols at the bottom right corner, including a plus sign and some illegible scribbles.

$$24. x^2 - 3bx + 2b^2 = 0.$$

$$25. x^2 + ax - cx = 0.$$

$$26. x^2 - nx + cx - cn = 0.$$

$$27. x^2 - nx = 5x - 5n.$$

$$28. (x - a)^2 - (2x - 1)(x - a) = 0.$$

$$29. (3x + 2)(x - 2n) - (x - 2n)^2 = 0.$$

$$30. x^2 - 4 - 3x + 6 = 0.$$

$$31. x^2 - a^2 - cx + ac = 0.$$

81. Cubic equations. An equation in x which may be put in the form

$$ax^3 + bx^2 + cx + d = 0,$$

where the coefficients a , b , c , and d represent numbers, is called a *cubic equation*, or an equation of the third degree.

Some equations of higher degree than the second may be solved by the method of factoring.

EXAMPLE

Solve the cubic equation $x^3 + x^2 = 4x + 4$.

Solution. Transposing, $x^3 + x^2 - 4x - 4 = 0$.

Factoring, $x^2(x + 1) - 4(x + 1) = (x + 1)(x^2 - 4)$
 $= (x + 1)(x - 2)(x + 2).$

Setting each factor equal to zero,

$$x + 1 = 0, \quad \text{therefore} \quad x = -1;$$

$$x - 2 = 0, \quad \text{therefore} \quad x = 2;$$

$$x + 2 = 0, \quad \text{therefore} \quad x = -2.$$

Check. When $x = -1$, $-1 + 1 = -4 + 4.$

When $x = 2$, $8 + 4 = 8 + 4.$

When $x = -2$, $-8 + 4 = -8 + 4.$

EXERCISES

Solve and check the following:

- | | |
|------------------------------|------------------------------------|
| 1. $x^3 - 9x = 0$. | 8. $x^3 + 2 = x + 2x^2$. |
| 2. $x^3 = 4x$. | 9. $y^3 + 4y^2 = 36 + 9y$. |
| 3. $x^3 - x^2 - 20x = 0$. | 10. $6x^2 - 9x = 54 - x^3$. |
| 4. $x^3 = x^2 + 12x$. | 11. $y^4 - 5y^2 + 4 = 0$. |
| 5. $x^3 + 6x = 5x^2$. | 12. $y^4 + 9 = 10y^2$. |
| 6. $2y^3 - 32y = y^2 - 16$. | 13. $y^4 + 36 = 13y^2$. |
| 7. $y^3 - 9y = 45 - 5y^2$. | 14. $x^3 - ax^2 = 4n^2x - 4an^2$. |

PROBLEMS

1. The square of a certain number plus the number itself equals 30. Find the number.

HINT. Translated into an equation this becomes $n^2 + n = 30$.

2. Four times the square of a certain number equals nine times the number. What is the number?

HINT. Translated into an equation this becomes $4n^2 = 9n$.

3. If to the square of a certain number the sum of twice the number and 7 be added, the result is 70. Find the number.

4. If from the square of a certain number twice the number be taken, the result is 24. Find the number.

5. A certain number is added to 19 and the same number is added to 25. The product of the two sums is 720. Find the number. $(x+19)(x+25)=720$

6. A certain number is subtracted from 17 and is also subtracted from 31. The product of the remainders is 240. Find the number. $(17-x)(31-x)=240$

7. The difference between two numbers is 7 and the difference of their squares is 203. Find the numbers.

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~~8.~~ A certain number is added to 23 and subtracted from 32. The product of the two results obtained is 92 more than 14 times the number. Find the number.
 $(23 + x)(32 - x) = 92 + 14x$

9. If from the square of three times a certain number 20 times the number be taken, the result will be 16 times the number. Find the number.
 $(3x)^2 - 20x = 16x$

10. The depth of a certain lot whose area is 4800 square feet is three times its frontage. Find the dimensions of the lot.
 $x \times 3x = 4800$

11. The area of the floor of a certain room is 80 square yards and the room is 6 feet longer than it is wide. Find the dimensions of the room.
 $7 + 6x = x \times 7x$

12. The area of a rectangular field is 80 square rods. The field is 11 yards longer than it is wide. Find its dimensions.
 $24 \times 20 = 480$
 $9 \times 21 = 189$

13. The sum of the squares of two consecutive numbers is 181. Find the numbers.

14. The sum of the squares of two consecutive odd numbers is 130. Find the numbers.

15. The difference of the squares of two consecutive even numbers is 76. Find the numbers.

16. The sum of the squares of three consecutive odd numbers is 155. Find the numbers.

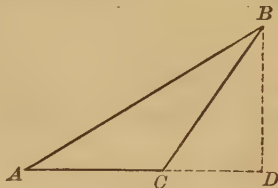
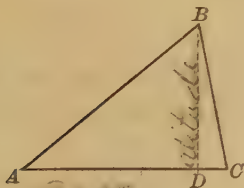
17. An uncovered square box 9 inches deep has 252 square inches of inside surface. Find its dimensions.
 $4 \times 9 \times 4 \times 9 = 252$

18. The entire outer surface of a cube is 150 square inches. Find the edge of the cube.
 $6x^2 = 150$

19. A rectangular box is four times as long and three times as wide as it is deep. There are 950 square inches in its entire outer surface. Find the dimensions.
 $6x^2 = 150$

20. A box is 5 inches longer and 3 inches wider than it is deep. There are 180 square inches in its entire outer surface. Find its dimensions.

The *altitude* of a triangle is the perpendicular from any vertex to the side opposite. This side is called the *base*.



In the adjacent figures, BD is the altitude and AC is the base of each triangle.

If a is the altitude of a triangle and b its base, the area of the triangle is $\frac{ab}{2}$.

$$\frac{1}{2} b \cdot a = \text{area}$$

In making use of this and similar formulas the unit in terms of which the lines are measured must be stated.

21. The area of a triangle is 42 square feet and the altitude is 7 feet. Find the base.

22. The altitude of a triangle is four times the base and the area is 50 square feet. Find the base and the altitude.

23. The base of a triangle is three times the altitude and the area is 54 square feet. Find the base and the altitude.

24. The area of a triangle is 144 square feet and the base is eight times the altitude. Find the base and the altitude.

25. The area of a triangle is 96 square feet and the base is 4 feet longer than the altitude. Find the base and the altitude.

HINT. Let

x = altitude in feet.

Then

$x + 4$ = base in feet,

and

$$\text{the area} = \frac{x(x + 4)}{2} = 96,$$

or

$$x^2 + 4x = 192, \text{ etc.}$$

26. The altitude of a triangle is 3 feet longer than the base. The area is 10 square yards. Find the base and the altitude.

27. In a triangle the two sides about the right angle differ by 5 feet. The area of the triangle is 150 square feet. Find the sides about the right angle.

28. The area of a triangle is 81 square feet and the altitude is twice the base. Find the base and the altitude.

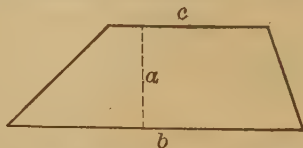
29. The area of a triangle is $6\frac{2}{3}$ square feet and the altitude is 28 inches longer than the base. Find the base and the altitude.

30. The sum of the two shorter sides of a right triangle is 34 feet and the area is 120 square feet. Find the base and the altitude of the triangle.

31. The area of a triangle is 12 square yards and the altitude is 3 feet longer than three times the base. Find the base and the altitude.

A *trapezoid* is a four-sided figure two of whose sides are unequal and parallel.

The bases of a trapezoid are the two parallel sides, b and c .



The altitude, a , is the perpendicular distance between the bases.

The area of a trapezoid is given by the formula $\frac{a(b + c)}{2}$.

32. Find the area of a trapezoid whose bases are 12 feet and 20 feet respectively, and whose altitude is 9 feet.

33. The altitude of a trapezoid is 10 inches, its area is 135 square inches, and one base is 5 inches longer than the other. Find the bases.

HINT. Let x = the length of one base in inches.

Then $x + 5$ = the length of the other base in inches,

and the area = $\frac{(x + x + 5)10}{2} = 135$,

or $20x + 50 = 270$, etc.

34. One base of a trapezoid is 10 feet, the other base is three times the altitude, and the area is 84 square feet. Find the altitude and the other base.

35. The altitude of a trapezoid is one half the shorter base, and the latter is two thirds of the other base. The area is 160 square feet. Find the bases and the altitude.

36. The bases of a trapezoid are respectively 4 feet and 8 feet longer than the altitude, and the area is 352 square feet. Find the bases and the altitude.

37. One base of a trapezoid is 14 feet longer than the other, and the altitude is one third the sum of the bases. The area is 54 square yards. Find the bases and the altitude.

38. The area of a trapezoid is 20 square yards, the altitude equals one base, and the other base exceeds the altitude by 6 feet. Find the bases and the altitude.

39. One base of a trapezoid exceeds the other by 12 feet, the altitude is 3 feet longer than five times the shorter base, and the area is 44 square yards. Find the bases.

REVIEW EXERCISES

1. Does $x = 7$ satisfy the equation $2x^2 + x - 66 = 0$? Does $x = -6$ satisfy it?
2. Give an example of (a) a linear equation, (b) a quadratic equation, (c) a cubic equation.
3. Is zero a root of the equation $x^3 + 5 - 4x = 5$? Is 2 a root? Is -2 a root?
4. Give an example of an equation of the second degree. Give one of the third degree.
5. Is 4 a root of the equation $\frac{30}{x-2} - 2x = 4$? Is 3 a root?
6. What conclusion can be drawn from the statement $25(7x - 35) = 0$? Explain.
7. Solve $2x^2 + 5x + 3 = 0$.
8. Solve $2x^2 + 5x = 88$.
9. Solve for x the equation $x^2 - 2ax + abx = 2a^2b$.
10. Given $t^3 - 64t = 0$ and $t^2 - 64 = 0$. If we solve the second equation, have we solved the first? Explain the point illustrated by these two equations.
11. If $abc = 0$, what conclusion regarding the values of a , b , or c can be drawn? What possible values may exist for a , b , and c ?
12. The product of two numbers whose difference is 4 equals 357. Find the numbers.
13. Four times the square of a certain number minus four times the number equals 399. Find the number.
14. A verbal problem apparently easily solved may present an impossible situation and lead to impossible results.

Note the following: A double-decked bus carried 25 passengers. Two more passengers rode inside than on top. Find the number of passengers riding inside.

15. Nine times the square of a certain number of books minus 15 times that number is 176. How many books are there? Is this result possible?

16. In Exercise 15 change the word "books" to "days." Are the results possible? Change it to "motor cars." Are the results possible?

17. The area of a square in square yards and its perimeter in feet are expressed by the same number. Find the dimensions.

18. Find two consecutive integers whose product is 210.

19. The product of a certain even number and the second odd number greater in value is 550. Find the odd and the even number.

20. The product of a certain odd number and the second greater even number is 208. Find the two numbers.

CHAPTER XVI

FRACTIONS

82. Algebraic fractions. In arithmetic the student learned the properties of such fractions as $\frac{2}{3}$, $\frac{3}{4}$, $\frac{7}{12}$, .2, .36, etc., and how to perform the four fundamental operations on such fractions. We must now study the properties of algebraic fractions and learn to perform the four fundamental operations on them.

The expression $\frac{a}{b}$, in which a and b represent numbers or polynomials, is an *algebraic fraction*. It is read " a divided by b ," or " a over b ." A fraction is an indicated quotient in which the dividend is the numerator and the divisor the denominator. The numerator and denominator are often called the *terms* of a fraction.

Certain operations upon fractions, such as multiplying both (numerator and denominator by a number (raising to higher terms), and dividing both numerator and denominator by a number (reducing to lower terms), are often necessary before the processes of addition or subtraction of two or more fractions can be performed.

The change of a fraction to lower or to higher terms, and the addition and the subtraction of fractions in both arithmetic and algebra, depend on the

PRINCIPLE. *The numerator and the denominator of a fraction may be multiplied by the same expression or divided by the same expression without changing the value of the fraction.*

Thus $\frac{4}{5} = \frac{4 \cdot 3}{5 \cdot 3} = \frac{12}{15}$ and $\frac{24}{42} = \frac{24 \div 6}{42 \div 6} = \frac{4}{7}$.

Similarly, $\frac{x}{z} = \frac{x \cdot a}{z \cdot a} = \frac{ax}{az}$ and $\frac{x \div n}{z \div n} = \frac{x/n}{z/n}$.

Since $\frac{3}{3}$, $\frac{6}{6}$, $\frac{a}{a}$, and $\frac{n}{n}$ are each equal to 1, each of the four preceding illustrations is really a multiplication or a division of a fraction by 1. This produces no change in the numerical value of *any* fraction, though it may change its form.

ORAL EXERCISES

What is the fraction obtained by multiplying the numerator and the denominator of each of the following fractions by the number on the right?

- | | | | |
|-----------------------|-----------------------|--------------------------|-------------------------------|
| 1. $\frac{1}{2}$, 3. | 3. $\frac{2}{5}$, 3. | 5. $\frac{3}{x}$, 5. | 7. $\frac{3}{n+2}$, a . |
| 2. $\frac{2}{3}$, 4. | 4. $\frac{3}{7}$, 4. | 6. $\frac{a}{n}$, c . | 8. $\frac{a-n}{a+n}$, $2n$. |

How is the second fraction obtained from the first?

- | | | |
|-------------------------------------|--------------------------------------|---|
| 9. $\frac{1}{3}$, $\frac{3}{9}$. | 11. $\frac{3}{5}$, $\frac{6}{10}$. | 13. $\frac{5}{9}$, $\frac{10a}{18a}$. |
| 10. $\frac{1}{4}$, $\frac{2}{8}$. | 12. $\frac{2}{7}$, $\frac{6}{21}$. | 14. $\frac{7}{8x}$, $\frac{14a}{16ax}$. |

What fractions are obtained by dividing both the numerator and the denominator of each of the following fractions by the number on the right?

- | | | | |
|-------------------------|--------------------------|----------------------------|---------------------------------|
| 15. $\frac{3}{6}$, 3. | 17. $\frac{12}{30}$, 6. | 19. $\frac{6n}{9}$, 3. | 21. $\frac{3n}{n+nx}$, n . |
| 16. $\frac{6}{15}$, 3. | 18. $\frac{15}{25}$, 5. | 20. $\frac{n}{nx}$, n . | 22. $\frac{n+an}{n-an}$, n . |

How is the second fraction obtained from the first?

$$23. \frac{6}{9}, \frac{2}{3}.$$

$$25. \frac{24}{32}, \frac{3}{4}.$$

$$27. \frac{nx^2}{n^2x}, \frac{x}{n}.$$

$$24. \frac{12}{21}, \frac{4}{7}.$$

$$26. \frac{n^2}{nx}, \frac{n}{x}.$$

$$28. \frac{3n^3}{9n^2y}, \frac{n}{3y}.$$

83. Reduction of fractions to lowest terms. A fraction is in its *lowest terms* when no factor except 1 is common to both numerator and denominator.

Cancellation is the process of dividing the numerator and the denominator of a fraction by a factor common to both.

EXAMPLES

Reduce to lowest terms:

$$1. \frac{63 a^3 b^2 c^2}{84 a b^2 c^4}.$$

$$\text{Solution. } \frac{63 a^3 b^2 c^2}{84 a b^2 c^4} = \frac{\cancel{3}^1 \cdot \cancel{7}^1 \cdot a^3 b^2 c^2}{2^2 \cdot \cancel{3}^1 \cdot \cancel{7}^1 \cdot a b^2 c^4} = \frac{3 a^2}{4 c^2}.$$

$$2. \frac{4 a^2 n - 36 n}{3 a^2 n - 18 a n + 27 n} = \frac{4 n(a+3)(\cancel{a-3})}{3 n(a-3)(\cancel{a-3})} = \frac{4 a + 12}{3 a - 9}.$$

The pupil should note that a factor which occurs one or more times in both numerator and denominator of a fraction can be canceled only the same number of times from each.

For reducing a fraction to its lowest terms we have the

RULE. Separate the numerator and the denominator into their *prime factors* and cancel the factors common to both.

learn

Cancellation as used in the rule means an actual division of the numerator and the denominator by the same expression. Therefore *only factors which are common to the numerator and the denominator can be canceled.*

The terms (the parts connected by plus or minus signs) in polynomial numerators and denominators, even if alike, can never be canceled. For example, it would be incorrect to "cancel" thus: $\frac{7 + x}{11 + x}$, as the resulting fraction would be $\frac{7}{11}$ instead of the true value, $\frac{10}{14}$, or $\frac{5}{7}$. Similarly, in the fraction $\frac{a + z + 3t^2}{x + z + 6t^2}$ no cancellation is possible.

We have seen that we may multiply or divide both numerator and denominator of a fraction by the same number without affecting the value of the fraction. But we should never forget that *adding the same number to or subtracting the same number from both numerator and denominator changes the value of the fraction.* Also, *squaring both numerator and denominator leads to a different value.* Compare this statement with the operations that may be performed on each member of an equation as given on pages 56-58.

ORAL EXERCISES

Reduce to lowest terms:

- | | | | |
|------------------------|--------------------------|-------------------------------|-----------------------------|
| 1. $\frac{18}{24}$ | 5. $\frac{10x^2}{25x^3}$ | 9. $\frac{60m^2}{90m^3}$ | 13. $\frac{36a}{96a^2}$ |
| 2. $\frac{27}{36}$ | 6. $\frac{14a}{42a^2}$ | 10. $\frac{\sqrt{36}a}{84ax}$ | 14. $\frac{121x}{77x^2}$ |
| 3. $\frac{12}{42}$ | 7. $\frac{8x}{56x^3}$ | 11. $\frac{18ax}{72x^2}$ | 15. $\frac{144x^2}{120x}$ |
| 4. $\frac{3ax}{15a^2}$ | 8. $\frac{9a^2}{54a}$ | 12. $\frac{24ax}{60a^2x}$ | 16. $\frac{108am}{144a^2m}$ |

17. $\frac{66 \, xn^2}{110 \, an}$. 20. $\frac{132 \, x^4}{144 \, x^2}$. 23. $\frac{-14}{-21}$. 26. $\frac{-363 \, x^5 y^2}{-33 \, x^3 y^3}$.
 18. $\frac{55 \, az}{132 \, a^2 z}$. 21. $\frac{-13}{39}$. 24. $\frac{3 \, a^2 x}{-36 \, ax^2}$. 27. $\frac{-45 \, h^3 k^2}{30 \, hk^2}$.
 19. $\frac{64 \, m^2}{128 \, m^3}$. 22. $\frac{-12}{18}$. 25. $\frac{-14 \, m^3 n^2}{120 \, m^3 n^3}$. 28. $\frac{160 \, s^2 t^3}{25 \, st^4 u}$.

EXERCISES

Reduce to lowest terms :

1. $\frac{32}{72}$. 7. $\frac{15 \, an^3}{25 \, a^3 n^2 x}$. 13. $\frac{14 \, n}{21 \, n^2 + 14 \, n}$.
 2. $\frac{54}{81}$. 8. $\frac{12 \, x^2 n^6}{18 \, xn^7}$. 14. $\frac{3 \, n - 2}{6 \, n^2 - 4 \, n}$.
 3. $\frac{a^3 b}{a^2 b^4}$. 9. $\frac{42 \, a^4 n^4 c}{56 \, a^3 n^6 c^2}$. 15. $\frac{x^2 - 4}{(x - 2)^2}$.
 4. $\frac{x^3 n}{xn^3}$. 10. $\frac{3 \, a}{3 \, a + 3}$. 16. $\frac{n^2 - 2 \, n + 1}{n^2 - 1}$.
 5. $\frac{an^3}{3 \, a^2 n}$. 11. $\frac{4 \, n - 6 \, n^2}{2 \, n^3}$. 17. $\frac{n^3 - 4 \, n}{(n + 2)^2}$.
 6. $\frac{10 \, an^2 x}{5 \, ax^2}$. 12. $\frac{3 \, nx + 3 \, n^2}{3 \, n^4}$. 18. $\frac{n^2 + 3 \, n - 10}{n^2 + 4 \, n - 5}$.
 19. $\frac{x^2 + 7 \, x - 18}{x^2 - 3 \, x + 2}$. 23. $\frac{3 \, n^2 - 3 \, n - 270}{3 \, n^2 - 243}$.
 20. $\frac{(x^2 - n^2)(x + 3 \, n)}{(x + n)^2(2 \, x + 3 \, n)}$. 24. $\frac{x^4 - 16}{x^3 - 2 \, x^2 + 4 \, x - 8}$.
 21. $\frac{n^2 - 25}{n^2 - n - 30}$. 25. $\frac{n^4 - a^4}{n^4 + 3 \, n^2 a^2 + 2 \, a^4}$.
 22. $\frac{x^2 - 5 \, xy + 4 \, y^2}{x^2 - 16 \, y^2}$. 26. $\frac{x^2 - 5 \, x + 6}{2 \, x^2 - 12 \, x + 18}$.

$$27. \frac{3a^2 - 6an + 3n^2}{a^2 + 6an - 7n^2}.$$

$$29. \frac{a^2 - (b - c)^2}{an + bn - cn}.$$

$$28. \frac{(x - 3)^2 - a^2}{2x - 6 + 2a}.$$

$$30. \frac{4n^3 - 5n^2 - 4n + 5}{8n^4 - 10n^3 + 12n - 15}.$$

Supply the missing terms in the following expressions :

$$31. \frac{3}{4} = \frac{?}{12}. \quad 32. \frac{15}{16} = \frac{?}{48}. \quad 33. \frac{9}{10} = \frac{18}{?}. \quad 34. \frac{12}{32} = \frac{6}{?} = \frac{?}{8}.$$

Change the following to equivalent fractions having as denominators the expressions given at the right :

$$35. \frac{12n}{m}, 12m^2n.$$

$$37. \frac{m+n}{m-n}, m^2 - n^2.$$

$$36. \frac{ab}{c}, abc.$$

$$38. \frac{5a+b}{3a+2b}, 15a^2 + 13ab + 2b^2.$$

84. Lowest common multiple. The *lowest common multiple* (L.C.M.) of two or more arithmetical or algebraic expressions is the expression having the least number of factors which will exactly contain each of the given expressions.

If two or more polynomials have no common factor other than 1, they are said to be *prime* to each other.

The L.C.M. of two prime expressions is their product.

Thus $7x^2z$ and $11at^2$ are prime to each other, as also are $3x^2 - 5x$ and $x^2 - 16$. But $a^2 - 16$ and $a^2 - 5a + 4$ are not prime to each other, since each contains the factor $a - 4$.

ORAL EXERCISES

Determine by inspection the L.C.M. of :

$$1. 4, 10. \quad 4. 6, 10. \quad 7. 10, 25. \quad 10. 4, 12.$$

$$2. 6, 8. \quad 5. 4, 10. \quad 8. 4, 14. \quad 11. 6, 12.$$

$$3. 6, 9. \quad 6. 10, 15. \quad 9. 7, 14. \quad 12. 8, 12.$$

13. 10, 12. *60*

16. 2, 5, 15. *30*

19. 4, 6, 8. *24*

14. 2, 4, 6. *12*

17. 3, 5, 10. *30*

20. 6, 8, 12. *24*

15. 3, 6, 9. *18*

18. 6, 5, 10. *30*

21. 4, 9, 12. *36*

22. Give one other common multiple for the numbers in Exercises 1-10.

EXAMPLE

Find the L.C.M. of $18x^2y$, $15xy^2$, $20x^3y^4$.

Solution.

$$18x^2y = 2 \cdot 3^2 \cdot x^2 \cdot y,$$

$$15xy^2 = 3 \cdot 5 \cdot x \cdot y^2,$$

$$20x^3y^4 = 2^2 \cdot 5 \cdot x^3 \cdot y^4.$$

Since the L.C.M. must contain each of the expressions, it must contain as factors 2^2 , 3^2 , 5, x^3 , and y^4 . If it does this it will contain the other factors, 2, 3, 5, x^2 ; y , x , and y^2 , and hence will contain any of the above numbers.

Therefore the L.C.M. = $2^2 \cdot 3^2 \cdot 5 \cdot x^3 \cdot y^4 = 180x^3y^4$.

The method of finding the L.C.M. of two or more expressions is stated in the following

RULE. *Separate each expression into its prime factors. Then find the product of all the different prime factors, using each factor the greatest number of times it occurs in any one expression.*

EXERCISES

Find the L.C.M. of:

1. 18, 24, 36.

4. 54, 75, 96.

7. n^3x , nx^3 , n^2x^2 .

2. 25, 30, 35.

5. n^2x , n^2x^2 , nx^2 .

8. $4n^2$, $6nx^2$, $9n^3x$.

3. 24, 36, 44.

6. an , a^2 , n^2 .

9. $9n^2$, $6n^3$, $12n^5$.

10. $3 ax, 5 a^2x, 7 a.$

15. $12 an, 3 an^2 - 3 a^2n.$

11. $3 a, 4 b, 6 nx.$

16. $a^2 - an, a^2n.$

12. $5 x^2y, 10 y^2z, 4 x^2yz^2.$

17. $nx^2, 3 n^2x, 9 an^2 - 6 a^2n.$

13. $36 ax^2, 42 axy, 63 x^2y.$

18. $cx + cy, dx + dy.$

14. $4 n, n^2 - bn.$

19. $3 n + 3 x, 6 an + 6 ax.$

20. $n^2 - nx, 3 an - 3 ax.$

21. $ax^2 - 9 a, x^2 - 5 x + 6, x^2 - 4 x + 4.$

Solution. $ax^2 - 9 a = a(x - 3)(x + 3).$

$x^2 - 5 x + 6 = (x - 2)(x - 3).$

$x^2 - 4 x + 4 = (x - 2)(x - 2).$

Therefore the L.C.M. = $a(x - 3)(x + 3)(x - 2)^2.$

22. $x^2 - 4, x^2 - x - 6.$

23. $x^2 - ax, x^2 - a^2.$

24. $n^2 - 4, n^2 - 8 n - 20.$

25. $2 n^3 - 2 n, 3 n^4 + 15 n^3 - 18 n^2.$

26. $a^2 + ab - 2 b^2, ac - 2 bd + 2 ad - bc.$

27. $a^3 - a, a^3 - a^2 - 2 a, a^2 - 2 a.$

85. Equivalent fractions. Two fractions are equivalent when one can be obtained from the other either by multiplying or by dividing both numerator and denominator by the same expression.

For example, $\frac{2}{3}$ and $\frac{10}{15}$ are equivalent fractions; also $\frac{nx}{n^2}$ and $\frac{x}{n}$.

The *lowest common denominator* (L.C.D.) of two or more fractions is the L.C.M. of their denominators.

$$\begin{array}{r} 5 \overline{) 56} = 11 \text{ R } 1 \\ \underline{55} \\ 1 \\ \underline{11} \\ 0 \end{array}$$

Before adding or subtracting fractions it is necessary to find equivalent fractions having the L.C.D.

EXAMPLES

Reduce to respectively equivalent fractions having the lowest common denominator :

1. $\frac{2}{3}$, $\frac{3}{4}$, and $\frac{5}{9}$.

Solution. The L.C.M. of the denominators is 36. Multiplying the numerator and denominator of the first fraction by 12, of the second by 9, and of the third by 4, we obtain $\frac{24}{36}$, $\frac{27}{36}$, and $\frac{20}{36}$ respectively.

2. $\frac{3a}{4n^2x}$ and $\frac{5n}{6ax^2}$.

Solution. The L.C.M. of the denominators is $12an^2x^2$. By inspection it is seen that $4n^2x$ multiplied by $3ax$ gives $12an^2x^2$, and $6ax^2$ multiplied by $2n^2$ gives $12an^2x^2$. Multiplying the numerator and the denominator of the first fraction by $3ax$ and of the second by $2n^2$ gives $\frac{9a^2x}{12an^2x^2}$ and $\frac{10n^3}{12an^2x^2}$ respectively.

Thus,
$$\frac{3a}{4n^2x} = \frac{3a \cdot 3ax}{4n^2x \cdot 3ax} = \frac{9a^2x}{12an^2x^2}$$

and
$$\frac{5n}{6ax^2} = \frac{5n \cdot 2n^2}{6ax^2 \cdot 2n^2} = \frac{10n^3}{12an^2x^2}.$$

Therefore, to change two or more fractions (in their lowest terms) to respectively equivalent fractions we have the

RULE. *Write the fractions with their denominators in factored form if they are not already so expressed.*

Find the L.C.M. of the denominators of the fractions.

Multiply the numerator and the denominator of each fraction by those factors of this L.C.M. which are not found in the denominator of that fraction.

ORAL EXERCISES

Transform to respectively equivalent fractions having the lowest common denominator:

- | | | | |
|-----------------------------------|-------------------------------------|------------------------------------|---|
| 1. $\frac{1}{2}, \frac{1}{4}.$ | 6. $\frac{1}{8}, \frac{1}{12}.$ | 11. $\frac{2}{3}, \frac{3}{5}.$ | 16. $\frac{3}{5}, \frac{4}{9}.$ |
| 2. $\frac{1}{6}, \frac{1}{3}.$ | 7. $\frac{1}{6}, \frac{1}{9}.$ | 12. $\frac{1}{6}, \frac{1}{8}.$ | 17. $\frac{1}{2}, \frac{1}{3}, \frac{1}{5}.$ |
| 3. $\frac{1}{2}, \frac{1}{6}.$ | 8. $\frac{1}{3}, \frac{1}{5}.$ | 13. $\frac{1}{9}, \frac{1}{12}.$ | 18. $\frac{1}{3}, \frac{1}{2}, \frac{1}{9}.$ |
| 4. $\frac{1}{3}, \frac{1}{4}.$ | 9. $\frac{1}{4}, \frac{1}{5}.$ | 14. $\frac{1}{10}, \frac{1}{15}.$ | 19. $\frac{1}{4}, \frac{1}{6}, \frac{1}{12}.$ |
| 5. $\frac{1}{4}, \frac{1}{16}.$ | 10. $\frac{2}{3}, \frac{1}{2}.$ | 15. $\frac{3}{10}, \frac{2}{5}.$ | 20. $\frac{2}{3}, \frac{3}{4}, \frac{1}{5}.$ |
| 21. $\frac{1}{a}, \frac{1}{a^2}.$ | 24. $\frac{3}{n^2}, \frac{2}{n^3}.$ | 27. $\frac{4}{an}, \frac{5}{n^2}.$ | |
| 22. $\frac{1}{ax}, \frac{1}{cx}.$ | 25. $\frac{1}{a}, \frac{1}{b}.$ | 28. $\frac{1}{b}, \frac{c}{d}.$ | |
| 23. $\frac{2}{x^2}, \frac{3}{x}.$ | 26. $\frac{2}{a}, \frac{3}{ab}.$ | 29. $\frac{a}{b}, \frac{c}{d}.$ | |

EXERCISES

Transform to respectively equivalent fractions having the lowest common denominator:

- | | | | |
|--|--|--------------------------------|---|
| 1. $\frac{a}{2}, \frac{x}{3}.$ | 3. $\frac{2n}{5}, \frac{3an}{4}, \frac{a^2}{6}.$ | 5. $\frac{a}{x}, \frac{x}{a}.$ | 7. $\frac{a}{b}, \frac{c}{d}, \frac{x}{y}.$ |
| 2. $\frac{2x}{3}, \frac{3a}{2}.$ | 4. $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}.$ | 6. $\frac{a}{n}, \frac{t}{x}.$ | 8. $\frac{a}{x^2}, \frac{c}{x}, \frac{t}{3}.$ |
| 9. $\frac{3a}{2b^2c}, \frac{2b}{3a^2c},$ and $\frac{c}{6ab}.$ | 12. $\frac{b-c}{2a}, \frac{a+b}{3c}.$ | | |
| 10. $\frac{3y}{2n^3z}, \frac{9x}{4ntz},$ and $\frac{7xy}{5nt^2z}.$ | 13. $\frac{2+x^2}{2x}, \frac{4+x^2}{4n}.$ | | |
| 11. $\frac{a+b}{c}, \frac{a-b}{a}.$ | 14. $\frac{a-x}{mn}, \frac{am-x}{n^2}.$ | | |

$$15. \frac{2}{a+3}, \frac{5}{a-3}.$$

Solution. By inspection it is seen that the L. C. D. is the product of the two denominators, or $(a+3)(a-3)$. Hence the multiplier for the numerator and denominator of the first fraction is $a-3$, and for the second is $a+3$. Using these, we obtain

$$\frac{2}{a+3} = \frac{2a-6}{a^2-9} \text{ and } \frac{5}{a-3} = \frac{5a+15}{a^2-9}.$$

$$16. \frac{1}{a+2}, \frac{2}{a-2}.$$

$$18. \frac{a}{a+n}, \frac{n}{a-n}.$$

$$17. \frac{x}{x-3}, \frac{2x}{x+3}.$$

$$19. \frac{x+2}{x-2}, \frac{x-2}{x+2}.$$

$$20. \frac{x+1}{x-2}, \frac{x-2}{x+1}.$$

$$21. \frac{a}{x+2}, \frac{3}{x-2}, \frac{a}{x}.$$

HINT. The L. C. D. = $x(x-2)(x+2)$. Hence the numerator and denominator of the first fraction must be multiplied by $x(x-2)$, of the second by $x(x+2)$, and of the third by $(x+2)(x-2)$.

$$22. \frac{1}{a}, \frac{2}{b}, \frac{3}{a+b}.$$

$$27. \frac{3x}{x^2-4}, \frac{5x}{6+3x}, \frac{x}{x-2}.$$

$$23. \frac{a}{x}, \frac{b}{ax}, \frac{c}{a+x}.$$

$$28. \frac{2x}{x-3}, \frac{5}{x^2-9}, \frac{5}{x^2-5x+6}.$$

$$24. \frac{2}{x+3}, \frac{3}{2x+6}.$$

$$29. \frac{x+3}{x^2-25}, \frac{x+1}{x^2-6x+5}, \frac{2}{x+5}.$$

$$25. \frac{n}{ax+a^2}, \frac{2}{x+a}.$$

$$30. \frac{1}{4a^2-1}, \frac{2a-1}{2a+1}, \frac{3}{a}.$$

$$26. \frac{a}{x^2-n^2}, \frac{b}{x+n}.$$

$$31. \frac{x+1}{2x+3}, \frac{3x}{2x^2+5x+3}.$$

NOTE. The problem of operating with fractions presented great difficulties to all the early races. The Egyptians and the Greeks, even down to the sixth century of our era, always reduced their fractions to the sum of several fractions each of which had 1 for a numerator. For example, $\frac{5}{8}$ would be expressed as $\frac{1}{2} + \frac{1}{8}$. The Romans usually expressed all the fractions of a sum in terms of fractions with the common denominator 12. The Babylonians resorted to a similar device, but used 60 for the denominator. In some way they all attempted to evade the difficulty of considering changes in both numerator and denominator. The Hindus seem to have been the first to reduce fractions to a common denominator, though Euclid (300 B. C.) was familiar with the method of finding the least common multiple of two or more numbers.

86. Addition and subtraction of fractions. If two or more fractions have the same denominator, their sum is the fraction obtained by adding their numerators and writing the result over their common denominator.

For example, $\frac{3}{7} + \frac{2}{7} + \frac{6}{7} = \frac{11}{7}$, and $\frac{n}{d} + \frac{3n}{d} + \frac{5n}{d} = \frac{9n}{d}$.

If two fractions have the same denominators, their difference is the fraction obtained by subtracting the numerator of the subtrahend from the numerator of the minuend and writing the result over their common denominator.

For example, $\frac{6}{11} - \frac{2}{11} = \frac{4}{11}$, and $\frac{a}{x} - \frac{c}{x} = \frac{a-c}{x}$.

If it is required to add or to subtract two fractions having unlike denominators, the fractions must be changed to respectively equivalent fractions having a common denominator; then their sum or their difference is obtained as explained above.

For example, to find the sum of $\frac{2}{3} + \frac{3}{5} + \frac{5}{6}$ we reduce the fractions to respectively equivalent fractions having the

common denominator 30, by multiplying both numerator and denominator of $\frac{2}{3}$ by 10, of $\frac{3}{5}$ by 6, and of $\frac{5}{6}$ by 5. The fractions become $\frac{20}{30}$, $\frac{18}{30}$, and $\frac{25}{30}$ respectively, and their sum is $\frac{63}{30}$, or $\frac{21}{10}$.

The pupil should always reduce fractional results to their lowest terms.

ORAL EXERCISES

Find the algebraic sum of :

1. $\frac{1}{3} + \frac{4}{3}$.

7. $\frac{1}{n} + \frac{2}{n}$.

13. $\frac{a+c}{ax} - \frac{c}{ax}$.

2. $\frac{4}{5} + \frac{6}{5}$.

8. $\frac{3}{n} + \frac{a}{n}$.

14. $\frac{a-n}{3x} - \frac{a+n}{3x}$.

3. $\frac{4}{7} + \frac{5}{7}$.

9. $\frac{10}{a} - \frac{3}{a}$.

15. $\frac{a^2+n^2}{a+n} - \frac{n^2-a^2}{a+n}$.

4. $\frac{6}{11} - \frac{2}{11}$.

10. $\frac{n}{ax} - \frac{3}{ax}$.

16. $\frac{a^2-ax}{a+x} + \frac{2ax}{a+x}$.

5. $\frac{5}{12} - \frac{3}{12}$.

11. $\frac{n+1}{a} - \frac{2}{a}$.

17. $\frac{n^3-n^2}{n+1} + \frac{2n^2}{n+1}$.

6. $\frac{11}{18} + \frac{5}{18}$.

12. $\frac{a-3}{x} + \frac{5}{x}$.

18. $\frac{2a+n}{a+an} - \frac{2n+a}{a+an}$.

In adding or subtracting algebraic fractions with unlike denominators, as $\frac{a}{x}$ and $\frac{c}{z}$, we proceed in a similar way, as follows :

Multiply both terms of $\frac{a}{x}$ by z , and of $\frac{c}{z}$ by x . The fractions become $\frac{az}{xz}$ and $\frac{cx}{xz}$ respectively, the sum of which is $\frac{az+cx}{xz}$. Similarly, $\frac{x}{y} - \frac{n}{d} = \frac{xd}{yd} - \frac{ny}{yd}$, which equals $\frac{xd-ny}{yd}$.

EXAMPLE

Find the algebraic sum of $\frac{3x}{5n^2} + \frac{x-5}{15a} - \frac{2x-n}{3an}$.

Solution. The L. C. D. is $15an^2$.

$$\begin{aligned}\frac{3x}{5n^2} + \frac{x-5}{15a} - \frac{2x-n}{3an} &= \frac{3x \cdot 3a}{5n^2 \cdot 3a} + \frac{(x-5)n^2}{15a \cdot n^2} - \frac{(2x-n)5n}{3an \cdot 5n} \\ &= \frac{9ax}{15an^2} + \frac{n^2x - 5n^2}{15an^2} - \frac{10nx - 5n^2}{15an^2} \\ &= \frac{9ax + (n^2x - 5n^2) - (10nx - 5n^2)}{15an^2} \\ &= \frac{9ax + n^2x - 5n^2 - 10nx + 5n^2}{15an^2} \\ &= \frac{9ax + n^2x - 10nx}{15an^2}.\end{aligned}$$

Check. The above solution gives:

$$\frac{3x}{5n^2} + \frac{x-5}{15a} - \frac{2x-n}{3an} = \frac{9ax + n^2x - 10nx}{15an^2}.$$

Now let $x = 1$, $a = 2$, and $n = 3$, and we obtain

$$\begin{aligned}\frac{3}{45} + \frac{1-5}{30} - \frac{2-3}{18} &= \frac{18+9-30}{270} \\ \frac{6-12+5}{90} &= \frac{-3}{270}, \text{ or } \frac{-1}{90} = \frac{-1}{90}.\end{aligned}$$

Therefore, to find the algebraic sum of two or more fractions (in their lowest terms) we have the

RULE. Reduce the fractions to respectively equivalent fractions having the lowest common denominator. Write in succession over the lowest common denominator the numerators of the equivalent fractions, inclosing each numerator in a parenthesis preceded by the sign of the corresponding fraction.

Rewrite the fraction just obtained, removing the parentheses in the numerator.

Then combine like terms in the numerator and, if necessary, reduce the resulting fraction to its lowest terms.

EXERCISES

Find the algebraic sum of :

$$1. \frac{2}{3} + \frac{3}{5}. \quad 3. \frac{7}{16} + \frac{5}{12}. \quad 5. \frac{4}{5} + \frac{5}{11}. \quad 7. \frac{5}{26} + \frac{8}{39} - \frac{17}{52}.$$

$$2. \frac{5}{6} + \frac{3}{8}. \quad 4. \frac{5}{21} + \frac{3}{14}. \quad 6. \frac{3}{4} + \frac{1}{3} + \frac{7}{15}. \quad 8. \frac{5}{6} + \frac{7}{12} - \frac{4}{9}.$$

$$9. \frac{5n}{3} + \frac{6n}{5}.$$

$$16. \frac{a}{x} - \frac{x}{a} + \frac{2}{ax}.$$

$$10. \frac{3n}{5} + \frac{2n}{3} + \frac{4n}{15}.$$

$$17. \frac{2}{3x} + \frac{5}{2x^2} - \frac{10x}{4x^3}.$$

$$11. \frac{x+3}{6} - \frac{3x+5}{8}.$$

$$18. \frac{a+b}{a} - \frac{2b}{ab} - \frac{a}{b}.$$

$$12. \frac{5n-4}{12} - \frac{2n+8}{15}.$$

$$19. \frac{a}{b} + \frac{b}{c} - \frac{c}{a}.$$

$$13. \frac{3x-2}{4} - \frac{5-x}{6} + \frac{5x}{9}.$$

$$20. \frac{3}{n^2} + \frac{4}{n} - \frac{8}{3n^3}.$$

$$14. \frac{1}{a} + \frac{1}{b}.$$

$$21. \frac{7}{a^2} - \frac{51}{ax} + \frac{6}{x^2}.$$

$$15. \frac{1}{a} + \frac{1}{b} - \frac{1}{c}.$$

$$22. \frac{3}{ab} - \frac{5}{2a^2} - \frac{6}{5ab^3}.$$

$$23. \frac{m}{n} - \frac{5}{2n^3} + \frac{x}{4}.$$

$$24. \frac{5n^3-8}{9n^4} - \frac{7-2n^2}{4n^3} + \frac{3n-1}{6n^2}.$$

$$25. \frac{n^2-4}{nx^2} - \frac{4n^2-9}{2nx} + \frac{6-n}{5n^2}.$$

$$26. \frac{3t-4}{5t^3} - \frac{5-3t}{2t} + \frac{4t^2-5}{3t^2}.$$

$$27. \frac{2n}{5n^2x} - \frac{3x-2}{10nx^2} - \frac{4nx+5}{15n^2x^2}.$$

$$28. \frac{x+4}{x^2-9} - \frac{2x-3}{x^2-5x+6}.$$

Solution.

$$\begin{aligned} \frac{x+4}{x^2-9} - \frac{2x-3}{x^2-5x+6} &= \frac{x+4}{(x-3)(x+3)} - \frac{2x-3}{(x-3)(x-2)} \\ &= \frac{(x+4)(x-2)}{(x-3)(x+3)(x-2)} - \frac{(2x-3)(x+3)}{(x-3)(x-2)(x+3)} \\ &= \frac{(x^2+2x-8) - (2x^2+3x-9)}{(x-3)(x-2)(x+3)} \\ &= \frac{x^2+2x-8-2x^2-3x+9}{(x-3)(x-2)(x+3)} \\ &= \frac{-x^2-x+1}{(x-3)(x-2)(x+3)} = \frac{1-x-x^2}{x^3-2x^2-9x+18}. \end{aligned}$$

(Unless otherwise directed the denominator should be retained in factored form throughout the solution.)

Check. Let $x = 4$.

$$\begin{aligned} \frac{x+4}{x^2-9} - \frac{2x-3}{x^2-5x+6} &= \frac{1-x-x^2}{x^3-2x^2-9x+18} \\ \frac{4+4}{16-9} - \frac{8-3}{16-20+6} &= \frac{1-4-16}{64-32-36+18} \\ \frac{8}{7} - \frac{5}{2} &= -\frac{19}{14}, \quad \text{or} \quad \frac{16}{14} - \frac{35}{14} = -\frac{19}{14}. \end{aligned}$$

In checking work in fractions we must assign such values to the letters as will make no denominator zero. This is necessary to avoid division by zero (see page 115). For this reason x in the above check cannot be 2, 3, or -3.

$$29. \frac{3}{x^2-4} - \frac{2}{x^2-3x+2}.$$

$$(31. \frac{2}{x-7} + \frac{3}{x+7}.$$

$$(30. \frac{3+n}{n-3} - \frac{4}{5}.$$

$$(32. \frac{4n}{n^2+nx} - \frac{11}{n}.$$

$$33. \frac{3x-y}{x^2-y^2} + \frac{5}{x-y}. \quad 37. \frac{8}{n^2-25} - \frac{1}{n^2-16n+55}.$$

$$34. \frac{9}{x^2-16} + \frac{5}{x^2-9x+20}. \quad 38. \frac{n+2}{n^2-25} - \frac{2n-1}{n^2-6n+5}.$$

$$35. \frac{3x}{a^2-ax} - \frac{a+2x}{a^2-x^2}. \quad 39. \frac{a-3}{a^2-6a} - \frac{2a+4}{a^2-8a+12}.$$

$$36. \frac{3n+2}{n+3} - \frac{5}{2n} + \frac{3}{4}. \quad 40. \frac{a^2-3ax+x^2}{a^2-6a+9} - \frac{3x-4a}{4a-12}.$$

$$41. \frac{n+3}{n^2+n} + \frac{2}{n} - \frac{5-n}{n^2+2n+1}.$$

$$42. \frac{a+x}{a^2-ax} - \frac{a-4x}{a^2-2ax+x^2} + \frac{3a-x}{a^2-x^2}.$$

$$43. \frac{2x+3}{x^2+3x-10} - \frac{3}{4x} + \frac{2-3x}{x^2+5x}.$$

$$44. \frac{x^2+3}{x^4-16} - \frac{2}{x^2-4} + \frac{3}{x^2+4}. \quad 45. \frac{x-a}{x+a} - \frac{x+a}{x-a} - \frac{3a}{x}.$$

87. Changes of sign in a fraction. The *sign of a fraction* is the plus or minus sign placed before the line separating the numerator from the denominator. Hence there are in a fraction three signs to consider: the sign of the fraction, the sign of the numerator, and the sign of the denominator.

Now in division the quotient of two expressions having like signs is positive, and the quotient of two expressions having unlike signs is negative.

Therefore

$$+ \frac{+12}{+3} = +4;$$

$$+ \frac{-12}{-3} = +4;$$

$$- \frac{-12}{+3} = -(-4) = +4;$$

$$- \frac{+12}{-3} = -(-4) = +4.$$

Or, in general terms, $+ \frac{+a}{+b} = + \frac{-a}{-b} = - \frac{-a}{+b} = - \frac{+a}{-b}.$

These examples illustrate the

PRINCIPLE. *Without altering the value of a fraction the following changes in sign may be made :*

- (a) The sign of the numerator and the sign of the denominator.
 (b) The sign of the numerator and the sign before the fraction.
 (c) The sign of the denominator and the sign before the fraction.*

Hence any fraction may be written in at least four ways, if proper changes of sign are made.

$$\text{Thus, } \frac{4a}{2-3x} = \frac{-4a}{3x-2} = -\frac{-4a}{2-3x} = -\frac{4a}{3x-2}.$$

Similarly,

$$\begin{aligned} \frac{a-2n}{3x-y+z} &= \frac{2n-a}{-3x+y-z} \\ &= -\frac{2n-a}{3x-y+z} = -\frac{a-2n}{-3x+y-z}. \end{aligned}$$

The pupil should note particularly that changing the sign of the numerator involves a change of sign in each term of the numerator. Similarly, a change of sign of the denominator involves a change of sign in each term of the denominator.

Multiplying one factor of an indicated product by -1 changes the sign of every term of the expanded product.

$$\text{Thus, } (n-3)(n-4) = n^2 - 7n + 12.$$

Multiplying the terms of the factor $n-3$ by -1 , we have

$$(3-n)(n-4) = -n^2 + 7n - 12.$$

Multiplying two factors of an indicated product by -1 does not change the sign of the expanded product.

$$\text{As before, } (n-3)(n-4) = n^2 - 7n + 12.$$

$$\begin{aligned} \text{But } (n-3)(-1)(n-4)(-1) &= (3-n)(4-n) \\ &= n^2 - 7n + 12. \end{aligned}$$

From these illustrations it follows that changing the sign of an *odd* number of factors in an indicated product changes the sign of every term of the expanded product, but if the sign of an *even* number of factors is changed the sign of the expanded product is not changed.

EXERCISES

Write as equivalent fractions in three other ways:

$$1. \frac{-n}{x}. \quad 3. -\frac{t}{n}. \quad 5. \frac{-2x}{a-b}. \quad 7. -\frac{a-b}{3x-2}.$$

$$\widehat{2.} \frac{a}{-x}. \quad 4. -\frac{3}{a-n}. \quad 6. \frac{3x-1}{2-5x}. \quad 8. \frac{a-x}{a^2-t^2}.$$

$$9. \frac{a-3}{x^2-2x-12}. \quad 10. -\frac{3-x}{2x-5}.$$

Write so that the letters are in alphabetical order in the factors of the denominator or so that the letter precedes the number if but one letter occurs in a factor:

$$11. -\frac{3}{(3-x)(x+4)}.$$

$$15. \frac{a-x}{(x-a)(x-c)}.$$

$$12. -\frac{4}{(a+b)(b-a)}.$$

$$16. \frac{-x-3}{(5-x)(-3-x)}.$$

$$13. \frac{-5}{(x+5)(7-x)}.$$

$$17. \frac{-7}{(a-b)(b-a)(c-a)}.$$

$$14. \frac{-6}{(a-c)(c-a)}.$$

$$18. \frac{-8}{(3-a)(b-a)(c-a)}.$$

Perform the indicated operation:

$$19. \frac{x-3}{x-5} - \frac{3x}{5-x}.$$

HINT. Rewrite so that the common denominator is $x-5$.

$$20. \frac{n}{x-n} - \frac{x}{n-x}.$$

$$21. \frac{n-1}{n-a} + \frac{3n-2}{a-n}.$$

$$22. \frac{a-1}{2-n} + \frac{3-a}{n-2} + \frac{2a}{2-n}.$$

$$23. \frac{x}{3-x} - \frac{x^2+4x}{x^2-9}.)$$

HINT. $\frac{x}{3-x} - \frac{x^2+4x}{(x-3)(x+3)} = \frac{-x}{x-3} - \frac{x^2+4x}{(x-3)(x+3)}$ etc.

$$24. \frac{3}{x^2-4} - \frac{5}{2-x}. \quad 26. \frac{x+4}{x^2-5x+6} - \frac{3}{3-x} + \frac{2}{2-x}.$$

$$25. \frac{x+1}{5-x} - \frac{x-1}{x^2-25}. \quad 27. \frac{x^2-tx}{x^2-4ax+3a^2} - \frac{x}{3a-x} + \frac{t}{a-x}.$$

$$28. \frac{t+3}{t^2-8t+15} - \frac{t}{5-t} - \frac{3}{3-t}.$$

$$29. \frac{2}{(a-b)(a-c)} - \frac{1}{(a-b)(c-a)}.$$

$$30. \frac{3n}{(n-3)(n-4)} - \frac{4n}{(3-n)(4-n)}.$$

$$31. \frac{6}{(a-7)(a-c)} - \frac{8}{(7-a)(c-a)}.$$

$$32. \frac{x-3}{x^2-5x+6} + \frac{2x-5}{9-x^2}.$$

$$33. \frac{3x-1}{x^3-9x} - \frac{3x-7}{x^2-6x+9}.$$

$$34. \frac{2a-3t}{a^2-5at+6t^2} - \frac{7t}{a^2-4t^2}.$$

$$35. \frac{x}{x^2-10x+24} - \frac{1}{6x-x^2-8} + \frac{3}{(6-x)(2-x)}.$$

88. Reduction of a mixed expression to a fraction. The mixed number $4\frac{2}{3}$ really means $4 + \frac{2}{3}$. It is equal to $\frac{4}{1} + \frac{2}{3}$, or $\frac{1 \cdot 2}{3} + \frac{2}{3} = \frac{1 \cdot 4}{3}$. The corresponding case in algebra is $a + \frac{b}{c}$.

$$\text{Now, } a + \frac{b}{c} = \frac{a}{1} + \frac{b}{c} = \frac{ac}{c} + \frac{b}{c} = \frac{ac + b}{c}.$$

Similarly,

$$\begin{aligned} n + 2 + \frac{3}{n-2} &= \frac{n+2}{1} + \frac{3}{n-2} = \frac{(n+2)(n-2)}{n-2} + \frac{3}{n-2} \\ &= \frac{n^2 - 4 + 3}{n-2} = \frac{n^2 - 1}{n-2}. \end{aligned}$$

The process involved in this operation is nothing more than the addition of two fractions, one of which has the denominator 1.

ORAL EXERCISES

Reduce to fractional form :

- | | | | |
|----------------------------|--|--------------------------------|--------------------------|
| 1. $2 + \frac{1}{3}$. | 4. $5 + \frac{3}{4}$. | 7. $3 + \frac{5}{12}$. | 10. $6 - \frac{2}{5}$. |
| 2. $3 + \frac{1}{5}$. | 5. $6 + \frac{5}{7}$. | 8. $4 - \frac{5}{6}$. | 11. $\frac{7}{5} - 2$. |
| 3. $4 + \frac{2}{3}$. | 6. $8 + \frac{3}{11}$. | 9. $5 - \frac{3}{5}$. | 12. $\frac{14}{3} - 3$. |
| 13. $1 + \frac{a}{x}$. | 19. $1 - \left(\frac{1}{2}\right)^2$. | 25. $4 - \frac{3}{n-2}$. | |
| 14. $2 - \frac{t}{a}$. | 20. $a^2 + \left(\frac{a}{2}\right)^2$. | 26. $x + 1 + \frac{1}{x-1}$. | |
| 15. $x - \frac{n}{x}$. | 21. $\frac{x^2}{4} + x$. | 27. $x - 2 + \frac{2}{x+2}$. | |
| 16. $n - \frac{3}{2x}$. | 22. $x^2 - \left(\frac{x}{2}\right)^2$. | 28. $x - 3 + \frac{1}{x+3}$. | |
| 17. $x + \frac{x}{n}$. | 23. $n^2 - \left(\frac{n}{3}\right)^2$. | 29. $n + 4 - \frac{16}{n-4}$. | |
| 18. $2a + \frac{3x}{2a}$. | 24. $3 + \frac{2}{x+1}$. | 30. $n - 5 - \frac{10}{n+5}$. | |

EXERCISES

Write as fractions and simplify results:

~~1.~~ $a + \frac{a}{x}$

~~5.~~ $a - 3 + \frac{4}{a}$

~~9.~~ $x - \frac{x+2}{x-2} + 1$

~~2.~~ $b - \frac{b}{b+1}$

~~6.~~ $2x + 3 - \frac{x^2}{2x}$

~~10.~~ $n - t + \frac{n^2 - t^2}{n+t}$

~~3.~~ $m + n + \frac{n}{m}$

~~7.~~ $b - \frac{6}{b+3} - 3$

~~11.~~ $\frac{a+2}{2a+1} + 2a - 1$

~~4.~~ $a + x - \frac{a}{x}$

~~8.~~ $n - 2 - \frac{2n}{n-3}$

~~12.~~ $\frac{x^2 + n^2}{3x - n} + 3x + n$

~~13.~~ $n - t + \frac{n^2 - 4t^2}{n+t}$

~~17.~~ $\frac{a}{a-2} - \frac{a-2}{a+2} - \frac{8}{a^2-4}$

~~14.~~ $2n - x - \frac{n^2 + x^2}{2n+x}$

~~18.~~ $\frac{b^2}{b^2-1} + \frac{b}{b+1} - \frac{b}{1-b}$

~~15.~~ $x^2 + x + 1 - \frac{x^3 + 2}{x-1}$

~~19.~~ $\frac{3a}{2+b} - \frac{a}{b-2} + \frac{1}{b^2-4}$

~~16.~~ $x^2 + y^2 - \frac{x^3 + y^3}{x+y} - xy$

~~20.~~ $\frac{x}{x+1} + \frac{2x^2}{x^2-1} - \frac{2x}{x-1}$

~~21.~~ $x^2 + x - \frac{x^4 + 3x^2 + 1}{x^2 - x + 1} + 1$

~~22.~~ $\left(2a + 3 - \frac{4}{a-2}\right) - \left(a + 1 - \frac{3}{a+4}\right)$

HINT. Removing parentheses,

$$2a + 3 - \frac{4}{a-2} - a - 1 + \frac{3}{a+4} = a + 2 - \frac{4}{a-2} + \frac{3}{a+4}, \text{ etc.}$$

~~23.~~ $\left(5x + \frac{6x}{a}\right) - \left(x - \frac{x-3a}{a}\right)$

~~24.~~ $\left(7x - \frac{2y}{3yz}\right) - \left(5x + \frac{x}{3yz}\right)$

$$25. \left(5n - \frac{2n}{x-y}\right) - \left(2n - \frac{4n}{x+y}\right).$$

$$26. \left(2n - 3t + \frac{3n}{2n+t}\right) + \left(3n + 2t - \frac{5n}{2n-t}\right).$$

$$27. \left(m + n - \frac{m}{m-n}\right) - \left(m - n + \frac{n}{m+n}\right).$$

$$28. \left(n - 3 - \frac{3n}{n+3}\right) - \left(n + 3 + \frac{3n}{n-3}\right).$$

$$29. \left(ax + a - \frac{a}{x+1}\right) - \left(ax - a - \frac{x}{x-1}\right).$$

89. Multiplication of fractions. In algebra as in arithmetic the product of two or more fractions is the product of their numerators divided by the product of their denominators.

Thus, $\frac{3}{5} \cdot \frac{4}{11} = \frac{12}{55}.$

Similarly, $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd},$

and $7 \cdot \frac{4}{5} = \frac{7}{1} \cdot \frac{4}{5} = \frac{28}{5}.$

In like manner, $n \cdot \frac{a}{b} = \frac{n}{1} \cdot \frac{a}{b} = \frac{na}{b}.$

If a factor occurs one or more times in any numerator and in any denominator of the indicated product of two or more fractions, it should be canceled the same number of times from both, thus giving the product of the several fractions in lower terms.

Thus, $\frac{a}{x} \cdot \frac{x}{n} = \frac{a}{n}.$

EXAMPLE

Multiply $\frac{3 x^2 y^3}{4 a^4}$ by $\frac{8 a^3}{12 x y}$.

$$\text{Solution. } \frac{3 x^2 y^3}{4 a^4} \cdot \frac{8 a^3}{12 x y} = \frac{\overset{xy^2}{\cancel{3} x^2 y^3}}{\underset{a}{\cancel{4} a^4}} \cdot \frac{\overset{2}{\cancel{8} a^3}}{\underset{2}{\cancel{12} x y}} = \frac{xy^2}{2 a}.$$

To find the product of two or more fractions or mixed expressions we have the

RULE. *If there are integral or mixed expressions, reduce them to fractional form.*

Separate each numerator and each denominator into its prime factors.

Cancel the factors common to any numerator and any denominator.

Write the product of the factors remaining in the numerator over the product of the factors remaining in the denominator.

ORAL EXERCISES

Find the product of

1. $\frac{1}{4} \cdot \frac{2}{3}$.

2. $6 \cdot \frac{3}{4}$.

3. $\frac{5}{6} \cdot \frac{18}{25}$.

4. $(\frac{4}{9})^2$.

5. $\frac{1}{a} \cdot \frac{1}{x}$.

10. $(\frac{n}{a^2})^2$.

15. $\frac{a^2 - 4}{3x} \cdot \frac{6x}{a - 2}$.

6. $\frac{a}{x} \cdot \frac{n}{c}$.

11. $2(\frac{a^3}{2n})^2$.

16. $\frac{3x + n}{nx} \cdot \frac{n^2 x^2}{3x - n}$.

7. $\frac{a}{2x} \cdot \frac{x}{a}$.

12. $2a(\frac{x}{a})$.

17. $(\frac{a + x}{a})^2 \cdot a$.

8. $\frac{a^2}{n^2} \cdot \frac{2n}{a}$.

13. $\frac{2n}{3} \cdot \frac{9}{4n}$.

18. $(\frac{n}{a + n})^2 (a + n)$.

9. $\frac{a^3}{n^2} \cdot \frac{n}{a}$.

14. $\frac{a^2 - x^2}{a} \cdot \frac{a}{a + x}$.

19. $\frac{n^2 - 9}{3x^2} \cdot \frac{x}{n - 3}$.

EXERCISES

Find the product of

1. $\frac{8}{9} \cdot \frac{12}{20}$.

4. $1\frac{3}{7} \cdot \frac{14}{5}$.

7. $1\frac{3}{7} \cdot 4\frac{1}{5}$.

2. $\frac{5}{21} \cdot \frac{14}{25}$.

5. $2 \cdot \frac{1}{5} \cdot \frac{15}{22} \cdot 6$.

8. $2\frac{3}{5} \cdot \frac{15}{26}$.

3. $\frac{4}{5} \cdot \frac{3}{8} \cdot 10$.

6. $1\frac{17}{18} \cdot 2\frac{4}{7}$.

9. $\frac{16}{35} \cdot 32 \cdot \frac{14}{24}$.

10. $\frac{10 a^2 n^2}{15 n^3} \cdot \frac{12 n}{7 a^3}$.

21. $\frac{x^4}{(2x)^3} \cdot \left(\frac{3a}{x}\right)^2 \cdot \left(\frac{2}{3}\right)^2$.

11. $\frac{6 an}{9 x} \cdot \frac{12 x^2 y}{16 n}$.

22. $\frac{6 n}{x} \left(\frac{-2x}{3n}\right)^3 \left(\frac{9n^2}{4x}\right)^2$.

12. $\frac{25 x^3}{6 a^2} \cdot \frac{4 a^2 x^4}{5 x^5}$.

23. $\frac{n+2}{n-2} \cdot \frac{n^2-4n+4}{n^2-4}$.

13. $\left(\frac{2x}{3a}\right)^2 \cdot \frac{9a}{x}$.

24. $\frac{a+x}{a-x} \cdot \frac{a^2-x^2}{a^2+2ax+x^2}$.

14. $\left(\frac{2ax}{5n}\right)^2 \cdot \frac{10n}{4a^3x}$.

25. $\frac{n^2-4}{n+3} \cdot \frac{2n+6}{3n-6}$.

15. $\frac{5ax}{2n^2} \cdot \frac{6nx}{10a^2} \cdot \frac{n^2a}{x^3}$.

26. $\frac{x+7}{x^2-25} \cdot \frac{3x-15}{ax+7a}$.

16. $\frac{a}{b} \cdot \frac{a^2x}{n^2} \cdot \frac{b^3n^3}{ax}$.

27. $\frac{3x-12}{ax+3a} \cdot \frac{a^2x+3a^2}{nx-4n}$.

17. $\frac{16a^2}{9n^4} \cdot \frac{12n^6}{40a^2} \cdot \frac{15na^3}{4n^3}$.

28. $\frac{5a+5c}{an-cn} \cdot \frac{an^2-cn^2}{a^2+ac}$.

18. $\left(\frac{2y}{3bz}\right)^2 \cdot \frac{12b^3}{16c^2yz^2} \cdot 8c^5z^3$.

29. $\frac{2n^2+6}{5x^4} \cdot \frac{10x^3}{3n^2+9}$.

19. $\frac{x}{4n^2} \left(\frac{n}{2x}\right)^2 \cdot 3\frac{1}{5}$.

30. $\frac{x^2-x-6}{x^2-4} \cdot \frac{x+2}{x-3}$.

20. $\left(\frac{1}{4}\right)^2 \cdot \left(\frac{2x}{n}\right)^3 \cdot \left(\frac{2n}{x}\right)^2$.

31. $\frac{x^2-16}{2x^2-18} \cdot \frac{x^2+x-6}{x^2+x-20}$.

$$32. \left(1 + \frac{2-2a}{a^2-1}\right) \left(\frac{2a}{a-1} - 1\right).$$

HINT. $\left(1 + \frac{2-2a}{a^2-1}\right) \left(\frac{2a}{a-1} - 1\right) = \frac{a^2-2a+1}{a^2-1} \cdot \frac{a+1}{a-1}$ etc.

$$33. \left(1 + \frac{3x}{1-2x}\right) \left(4 - \frac{3}{1-x^2}\right).$$

$$34. \left(\frac{6t-9}{2t-3} + 2t\right) \left(2t-9 + \frac{36}{2t+3}\right).$$

$$35. \frac{9n^3}{3n^2-54n+51} \cdot \left(1 + \frac{2}{n} - \frac{3}{n^2}\right).$$

$$36. \frac{ax^2+2ax+4a}{6xy} \cdot \frac{12x^2y^2}{3x^2+6x+12} \cdot \left(3 - \frac{4}{2-x}\right).$$

$$37. \left(\frac{1}{a} + \frac{1}{a^2} - \frac{30}{a^3}\right) \left(a + \frac{20-9a}{5-a} - 4\right).$$

$$38. \left(1 + \frac{1-7a}{a^2-1}\right) \left(\frac{5a^4+5a^3}{3a^2-33a+84}\right) \left(1 - \frac{5}{a} + \frac{4}{a^2}\right).$$

$$39. \left(n + 3 + \frac{n-3}{3n+5}\right) \left(4 - 3n - \frac{21}{n+4}\right) \left(\frac{1}{2n+2}\right).$$

90. Division of fractions. In arithmetic, $\frac{2}{5} \div \frac{4}{7} = \frac{2}{5} \cdot \frac{7}{4} = \frac{14}{20} = \frac{7}{10}$, and $\frac{3}{7} \div 8 = \frac{3}{7} \cdot \frac{1}{8} = \frac{3}{56}$. Also, $\frac{5}{7} \div 2\frac{3}{4} = \frac{5}{7} \div \frac{11}{4} = \frac{5}{7} \cdot \frac{4}{11} = \frac{20}{77}$.

Similarly, $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$, and $\frac{a}{b} \div n = \frac{a}{b} \cdot \frac{1}{n} = \frac{a}{bn}$.

Also, $\frac{a}{b} \div \left(c + \frac{n}{d}\right) = \frac{a}{b} \div \left(\frac{cd+n}{d}\right) = \frac{a}{b} \cdot \frac{d}{cd+n} = \frac{ad}{bcd+bn}$.

For division of fractions we have the

RULE. Reduce all integral or mixed expressions to fractional form.

Then invert the divisor, or divisors, and proceed as in multiplication of fractions.

NOTE. *Inverting the divisor.* The student of algebra will remember that in arithmetic the divisor is inverted in division of fractions, but he may fail to recall just why. It is worth while for anyone to understand thoroughly this particular step of the process of division of fractions and why division is then replaced by multiplication. In a book on arithmetic the explanation could be made clear by analyzing the solution of a simple problem like $8 \div \frac{3}{5}$. To see just why the divisor is here inverted consider first $1 \div \frac{3}{5}$. This last requires that a unit of one kind (unity itself) be divided by three units of a different kind (fifths). Any difficulty of this nature is avoided by reducing the number 1 to fifths; then $1 \div \frac{3}{5}$ becomes $\frac{5}{5} \div \frac{3}{5}$, and the problem is transformed to that of dividing three units of a certain kind (fifths) into five of the same kind. This can be done directly and we have $\frac{5}{5} \div \frac{3}{5} = \frac{5}{3}$. But the answer $\frac{5}{3}$ is $\frac{3}{5}$ inverted. This means that to find how often $\frac{3}{5}$ is contained in the number 1 we merely invert it and obtain $\frac{5}{3}$. Obviously $\frac{3}{5}$ is contained in 2 twice as often as in 1, in 3 three times as often, and in 8 eight times as often, etc. Hence $8 \div \frac{3}{5} = 8 \times \frac{5}{3} = \frac{40}{3}$.

In the general problem $n \div \frac{a}{c}$ we may as before write $1 \div \frac{a}{c} = \frac{c}{a} \div \frac{a}{c} = \frac{c}{a}$. That is, $\frac{a}{c}$ is contained in the number 1, $\frac{c}{a}$ times. In n it will be contained n times as often as in 1. Hence $n \div \frac{a}{c} = n \times \frac{c}{a} = \frac{nc}{a}$.

EXERCISES

Perform the indicated operations:

1. $\frac{3}{2} \div \frac{4}{7}$.

5. $\left(4 - \frac{3}{4}\right) \div \left(4 + \frac{7}{8}\right)$.

2. $\frac{5}{51} \div \frac{10}{17}$.

6. $\left(7 + \frac{4}{11}\right) \div \left(1 - \frac{13}{22}\right)$.

3. $\frac{23}{39} \div \frac{115}{26}$.

7. $\frac{a}{b} \div \frac{c}{d}$.

4. $\left(4 + \frac{1}{5}\right) \div \left(4 + \frac{9}{10}\right)$.

8. $\frac{a}{b} \div \frac{a^2}{bc}$.

$$9. \frac{3ab}{x} \div \frac{ac}{4nx}.$$

$$10. \frac{a^2}{n} \div \frac{2a}{n^2}.$$

$$11. \left(\frac{2an}{x}\right)^2 \div \frac{6an^2}{9x^4}.$$

$$12. \frac{a}{b} \div \frac{x}{n} \div \frac{an^2}{bx^2}.$$

$$13. \frac{3an}{4bx} \div \frac{6a^2n}{5b^2x} \div \frac{10an^2}{8cx}.$$

$$14. \frac{15}{4n^2} \div \frac{5a}{2n^6} \div \frac{(3n^2)^2}{12a^2}.$$

$$15. \frac{12y^3}{10x^2} \div \frac{6y}{5x} \div \frac{4ax}{3by^2}.$$

$$16. \frac{(4n^2)^2}{12} \cdot \left(\frac{2}{n}\right)^2 \div \frac{(2n)^2}{6}.$$

$$17. \frac{10ab^2}{21a^3} \div \frac{5b}{3a^2} \div 14a^4b.$$

$$18. \left(\frac{3a}{2x}\right)^2 \div \left(\frac{3a}{4x^2}\right)^2 \cdot \left(\frac{a}{x}\right).$$

HINT. See Rule, p. 20.

$$19. \frac{(10n^2)^2}{256} \div \left(\frac{5n^2}{2}\right)^2 \div \frac{5n^3}{(8n^4)^2}.$$

$$20. \frac{a^2}{x} \div \left(\frac{x}{a}\right)^3 \div \left(\frac{a}{x^2}\right)^3 \cdot \left(\frac{1}{x^2}\right)^3.$$

$$21. \frac{2n-1}{15ax^3} \div \frac{6n-3}{5x}.$$

$$22. \frac{1-x^2}{x^2-16} \div \frac{x-1}{x^2-7x+12}.$$

$$23. \frac{n^3-n^2-12n}{n^2-4n+4} \div \frac{3n^2+n^3}{4-n^2}.$$

$$24. \left(3 - \frac{1}{n+2}\right) \div \left(3 - \frac{4}{n+3}\right).$$

$$25. \left(x + \frac{n^2}{n+x}\right) \div \left(\frac{2nx+x^2}{n+x} - n\right).$$

$$26. \left(\frac{n}{a} - \frac{9a}{n}\right) \div \left(\frac{n^2}{4a^2} - \frac{81a^2}{4n^2}\right).$$

$$27. \left(x - 3 - \frac{28}{x}\right) \div \left(1 - \frac{1}{x} - \frac{20}{x^2}\right).$$

$$28. \left(n - \frac{3}{4n} + 1\right) \div \frac{2n+3}{2n+1} \left(\frac{3}{4n^2-1} + 3\right).$$

$$29. \left(\frac{2x-6}{x+2}\right) \div \left(3 + \frac{45}{4x^2-16}\right) \left(\frac{7}{x-3} + 2\right).$$

$$30. \left(\frac{a}{n} - \frac{n}{a}\right) \div \left(\frac{a+n}{2a^2-2an}\right) \cdot \frac{nx}{2(a-n)^2}.$$

$$31. \frac{9n^2 + 9n}{9n^2 - 4} \div \frac{n+1}{3n-2} \left(3n + 4 + \frac{4}{3n} \right).$$

$$32. \frac{4 - 4n^4}{4n^4 - 25n^2} \div \frac{2n^2 + 2}{2n - 5} \left(3n^2 + \frac{8n^2 - n^3}{n-1} \right).$$

$$33. (3n^3 - 75n) \div \left(2 + \frac{11}{n} + \frac{5}{n^2} \right) \left(6n - 11 - \frac{7}{n} \right).$$

$$34. \left(\frac{-6a}{a^2 - 4} + \frac{3}{2 - a} \right) \div \left(\frac{3}{a^2 - a - 6} \right).$$

$$35. \left(4 - \frac{6}{a+1} \right) \div \left(8 - \frac{4a-8}{a^2-1} \right).$$

$$36. \left(\frac{2x}{x-2} - \frac{x}{x-1} \right) \div \left(\frac{3x}{x-3} - \frac{2x}{x-2} \right).$$

$$37. \left(\frac{8}{m-x} - \frac{4}{x^2 - m^2} \right) \div \left(\frac{1}{m-x} - \frac{1}{m+x} \right).$$

$$38. \left(\frac{-5}{x^2 - 4} + \frac{2}{2-x} \right) \div \left(2 + \frac{3}{x-2} \right).$$

91. Complex fractions. A complex fraction is a fractional expression containing one or more fractions either in the numerator or in the denominator or in both.

EXAMPLE

$$\text{Simplify } \frac{\frac{16}{x} - x}{\frac{24}{x^3} + \frac{10}{x^2} + \frac{1}{x}}.$$

Solution. Reducing the numerator and denominator to simple fractions,

$$\frac{\frac{16}{x} - x}{\frac{24}{x^3} + \frac{10}{x^2} + \frac{1}{x}} = \frac{\frac{16 - x^2}{x}}{\frac{24 + 10x + x^2}{x^3}}.$$

Performing the indicated division,

$$\frac{\frac{16 - x^2}{x}}{24 + \frac{10x + x^2}{x^3}} = \frac{(4 - x)(4 + x)}{x} \cdot \frac{x^3}{(x + 6)(x + 4)} = \frac{4x^2 - x^3}{x + 6}.$$

Check. If we let $x = 2$,

$$\frac{\frac{16}{x} - x}{\frac{24}{x^3} + \frac{10}{x^2} + \frac{1}{x}} = \frac{4x^2 - x^3}{x + 6} \quad \text{becomes} \quad \frac{\frac{16}{2} - 2}{\frac{24}{8} + \frac{10}{4} + \frac{1}{2}} = \frac{4 \cdot 4 - 8}{2 + 6},$$

or $\frac{6}{6} = \frac{8}{8}.$

For simplifying a complex fraction we have the

RULE. Reduce the numerator and the denominator each to a simple fraction. Then multiply the numerator by the inverted denominator.

EXERCISES

Simplify:

1. $\frac{4 - \frac{1}{9}}{2 - \frac{1}{3}}.$

4. $\frac{2^3 - \left(\frac{1}{3}\right)^3}{3^2 + 1 + \left(\frac{1}{3}\right)^2}.$

7. $\frac{\frac{a}{b}}{\frac{b}{a} - \frac{a}{b}}.$

2. $\frac{2 + \frac{3}{5}}{4 - \frac{1}{10}}.$

5. $\frac{\left(\frac{2}{5}\right)^3 + \left(\frac{5}{3}\right)^3}{\left(\frac{2}{5}\right)^2 - \left(\frac{2}{5}\right)\left(\frac{5}{3}\right) + \left(\frac{5}{3}\right)^2}.$

8. $\frac{\frac{a}{b}}{\frac{c}{d}}.$

3. $\frac{\left(\frac{3}{5}\right)^2 - 1}{\frac{3}{5} + 1}.$

6. $\frac{\frac{4}{a}}{3 - \frac{1}{a}}.$

9. $\frac{1 - \frac{3}{a} - \frac{10}{a^2}}{a - 13 + \frac{40}{a}}.$

$$10. \frac{\frac{1+x}{x}}{1 - \frac{1}{x^2}}$$

$$12. \frac{\frac{y}{x} - \frac{8x}{y} - 2}{1 + \frac{y}{x} - \frac{20x}{y}}$$

$$14. \frac{\frac{1}{a^2} - \frac{1}{b^2}}{\frac{1}{a} + \frac{1}{b}}$$

$$11. \frac{\frac{a}{b}}{c} \div \frac{\frac{a}{b}}{\frac{b}{c}}$$

$$13. \frac{1 + \frac{x}{x-y}}{\frac{x}{x-y}}$$

$$15. \frac{x + \frac{1}{x+2} - 2}{x + \frac{1}{x-2} + 2}$$

$$16. \frac{\frac{(x-2a)^2}{4ax} + 2}{2 + \frac{x}{a}}$$

$$21. \frac{\frac{(x+5y)^2}{10xy} - 2}{\frac{1}{y} - \frac{5}{x}}$$

$$17. \frac{\frac{x}{1+x} - \frac{1-x}{x}}{\frac{x}{1+x} + \frac{1-x}{x}}$$

$$22. \frac{\frac{a}{3b} - \frac{3b}{a}}{\frac{4b}{a+b} - 1}$$

$$18. \frac{\frac{a^2+b^2}{b} - 2a}{1 - \frac{b}{a}}$$

$$23. \frac{10 + \frac{a}{x}}{\frac{(a-10x)^2}{20ax} + 2}$$

$$19. \frac{\frac{1}{2x} + \frac{1}{y}}{\frac{1}{5y^2} + \frac{1}{2xy} + \frac{1}{5x^2}}$$

$$24. \frac{a + \frac{t^2}{a-2t}}{\frac{2t^2}{a} - \frac{t}{a+t}}$$

$$20. \frac{\frac{1}{a^2} - \frac{1}{a^3} - \frac{12}{a^4}}{a - \frac{9}{a}}$$

$$25. \frac{8a - \frac{(3a+2x)^2}{3x}}{\left(\frac{3a+4x}{3a}\right)^2 - \frac{6x}{a}}$$

REVIEW EXERCISES

1. How are fractions added when they have the same denominators?
2. Is $\frac{5}{12} + \frac{7}{12} + \frac{1}{12}$ an operation of the type referred to in Exercise 1?
3. State an exercise of this type involving letters only.
4. In finding the sum of $\frac{3}{4} + \frac{4}{5}$, why is it not correct to add the denominators for the denominator of the result?
5. State the principle on which the changing of fractions to equivalent fractions in higher or lower terms depends.
6. What are equivalent fractions?
7. When we write $\frac{3}{5} + \frac{4}{7} = \frac{21}{35} + \frac{20}{35}$, on what fundamental principle is the equality based?
8. If 3 is added to the numerator and the denominator of $\frac{1}{15}$, will the value of the resulting fraction be equivalent to $\frac{1}{15}$? If 3 is subtracted?
9. If $\frac{a-3}{x-3}$ is transformed to $\frac{a}{x}$, what operations are really performed? Does this change the value of the fraction?
10. If $\frac{x+a}{3y+a}$ is changed to $\frac{x}{3y}$, what operations are really performed? Has the value of the fraction been changed?
11. Mention an error which must be avoided in simplifying the fraction

$$\frac{x+3+\frac{(x-2a)^2}{a}}{2x+4-\frac{x-2a}{a}}$$

12. Give an example of a mixed number; of a mixed expression.

13. Why is it desirable to write

$$\frac{1}{(a-x)(x-b)} + \frac{3}{(x-a)(b-x)}$$

in the form $\frac{-1}{(a-x)(b-x)} + \frac{-3}{(a-x)(b-x)}$

before adding? By the use of what principle is the second form of the fractions obtained?

14. State the arithmetic method of multiplying $\frac{3}{5}$ by $\frac{4}{7}$. Is the process the same for algebraic fractions?

15. State the arithmetic method of dividing 5 by $\frac{3}{7}$. Is the process the same for the division of algebraic fractions?

16. What is a complex fraction?

17. How can the numerator of a complex fraction be distinguished from the denominator? Consider

$$\frac{\frac{a}{b}}{\frac{c}{d}}$$

Which is the numerator? the denominator?

18. How is a complex fraction simplified?

19. In simplifying complex fractions like those on page 211, which of the four fundamental operations must be performed first? Which one is performed last?

20. In what two ways may $\frac{(\frac{2}{3})^2 - 1}{(\frac{2}{3}) + 1}$ be simplified?

21. Simplify $\frac{4 - (\frac{3}{5})^2}{2 + \frac{3}{5}}$ in the shortest way.

CHAPTER XVII

EQUATIONS CONTAINING FRACTIONS

92. Equations containing fractions with monomial denominators. If fractions are involved in one or both members of an equation, it is necessary to find a number or a literal expression by which one may multiply both members in order to get rid of the fractions. (Compare Example 1, p. 216.) This process involves the application of Axiom III, p. 57, which is the only principle employed in this chapter that has not been used repeatedly in the earlier work with equations.

Especial care is required to avoid errors when a fraction which has two or more terms in its numerator is preceded by a minus sign.

ORAL EXERCISES

Solve for x , stating what operations are necessary :

1. $\frac{x}{3} = 4.$

5. $\frac{2x}{3} = 4.$

9. $\frac{x}{5} - 7 = 0.$

2. $\frac{x}{5} = 8.$

6. $\frac{3x}{2} = 9.$

10. $\frac{x}{8} - 3 = 0.$

3. $\frac{x}{7} = 2.$

7. $\frac{3x}{5} = 6.$

11. $\frac{2x + 4}{5} = 0.$

4. $\frac{x}{10} = 3.$

8. $\frac{5x}{7} = 10.$

12. $\frac{3x - 6}{2} = 0.$

13. $\frac{3x-4}{2} = 0.$

19. $\frac{1}{x} = 2.$

25. $\frac{2}{x+1} = 1.$

14. $\frac{x+1}{2} = 3.$

20. $\frac{2}{x} = 1.$

26. $\frac{5}{x-1} = 1.$

15. $\frac{x-1}{3} = 1.$

21. $\frac{3}{x} = 5.$

27. $\frac{3}{x+1} = 4.$

16. $\frac{x-2}{4} = 5.$

22. $\frac{5}{x} = 8.$

28. $4 = \frac{1}{x-3}.$

17. $\frac{3x+2}{2} = 4.$

23. $\frac{3}{2x} = 5.$

29. $\frac{2}{x+1} = \frac{1}{x}.$

18. $\frac{5x+2}{3} = 4.$

24. $\frac{3}{5x} = 15.$

30. $\frac{a}{x} = \frac{x}{a}.$

31. Using the least multipliers possible, clear of fractions
Exercises 1-10 on page 215.

EXAMPLES

1. Solve the equation $\frac{2x}{5} - \frac{x}{7} = 9.$

Solution. Multiplying both members by 35 (L. C. M. of the denominators) and canceling, we obtain

$$14x - 5x = 315.$$

Then $12 \qquad 9x = 315,$

or $102 \qquad x = 35.$

Check. Substituting 35 for x in the given equation,

$$\frac{70}{5} - \frac{35}{7} = 9.$$

$$14 - 5 = 9.$$

2. Solve $\frac{x}{3} - \frac{3x}{5} \left(\frac{10}{x} - 4 \right) + 3\frac{3}{5} = 14.$

Solution. Removing the parenthesis,

$$\frac{x}{3} - 6 + \frac{12x}{5} + \frac{18}{5} = 14.$$

Collecting, $\frac{x}{3} + \frac{12x}{5} = \frac{82}{5}.$

Multiplying by 15 (the L. C. M. of the denominators) gives

$$5x + 36x = 246.$$

Then $41x = 246,$

or $x = 6.$

Check. $\frac{6}{3} - \frac{3 \cdot 6}{5} \left(\frac{10}{6} - 4 \right) + 3\frac{3}{5} = 14.$

$$2 - 6 + \frac{72}{5} + \frac{18}{5} = 14.$$

$$-4 + 18 = 14.$$

For solving equations containing fractions with monomial denominators, we have the

RULE. *Free the equation of any parentheses it may contain.*

Find the L. C. M. of the denominators of the fractions and multiply each term of the equation by it, using cancellation whenever possible.

Transpose and solve as usual.

EXERCISES

Solve and check:

1. $\frac{x}{2} + \frac{x}{5} = 14.$

4. $3x - \frac{5x}{7} = 8.$

2. $\frac{x}{3} + \frac{2x}{5} = 11.$

5. $\frac{x}{2} + \frac{x}{3} + \frac{x}{4} = 9.$

3. $\frac{2x}{5} + \frac{3x}{2} = 19.$

6. $\frac{1}{x} + \frac{2}{x} + \frac{3}{x} = 3.$

$$7. \frac{x+4}{2} - \frac{x-3}{5} = 5.$$

$$13. \frac{t-5}{t+4} = \frac{7}{16}.$$

$$8. \frac{3x+1}{4} + \frac{5x-1}{6} = 8.$$

$$14. \frac{2t-3}{3t-2} = \frac{17}{28}.$$

$$9. \frac{6}{t} - \frac{12+t}{3t} = \frac{4}{3}.$$

$$15. \frac{2}{t-3} = \frac{1}{t-5}.$$

$$10. \frac{t-2}{3t} + \frac{5}{6} - \frac{3t-5}{4t} = 0.$$

$$16. \frac{t-.6}{t+.4} = \frac{1}{6}.$$

$$11. 5t + \frac{1}{4} - 4t\left(3 - \frac{2}{t}\right) = 3.$$

$$17. \frac{.2t+4}{5} = \frac{.3t-3}{6}.$$

$$12. \frac{3+2t}{2t-3} = \frac{1}{4}.$$

$$18. \frac{t+5}{t-9} = \frac{t+4}{t-3}.$$

$$19. \frac{5t-3}{4} + \frac{3}{4}(t-1) = \frac{5}{2}.$$

$$20. \frac{1}{2}\left(4 - \frac{t}{2}\right) - \frac{3t-16}{6} = \frac{7-t}{6}.$$

$$21. \frac{t-3}{t-4} = \frac{t+5}{t+2}.$$

$$22. \frac{x+3}{x-5} = x-2.$$

$$26. \frac{5x+1}{x-11} + 8 = 2x.$$

$$23. \frac{x+5}{x-3} = x-4.$$

$$27. \frac{t+4}{6t-2} = \frac{t}{2} + \frac{3t+10}{6}.$$

$$24. \frac{x}{x+1} + \frac{1}{x-1} = \frac{x+3}{x+1}.$$

$$28. \frac{t+2}{t+3} + \frac{t+3}{t+2} = \frac{11t+28}{t^2+5t+6}.$$

$$25. 8 + \frac{x}{x-9} = x - \frac{7}{x-9}.$$

$$29. \frac{x}{n} + n = 1 + x.$$

$$30. \frac{3x}{6} - \frac{x-3}{3} = \frac{19}{3} - \frac{x+4}{2} + 10x.$$

$$31. \frac{4a}{5} + \frac{12(3a+1)}{2} = \left(\frac{a}{3} + 2\right) + 40 + \frac{14}{15}.$$

$$32. \frac{x}{2} - 2\left(\frac{4x}{5} - 3\right) = 4 - \frac{3}{2}\left(\frac{x}{2} + 1\right).$$

$$33. 4 - \frac{5x - 15}{4} + \frac{2(x + 2)}{3} = \frac{5(x - 1)}{6}.$$

$$34. \frac{x}{n} + \frac{x}{a} = a + n.$$

$$35. \frac{n}{x} + \frac{3n}{2x} = \frac{5}{4}.$$

$$37. \frac{x}{a} + \frac{x}{n} + an = bn + ab + \frac{x}{b}.$$

$$36. \frac{2cx}{n} - n^2 + 4c^2 = \frac{nx}{2c}.$$

$$38. \frac{x}{n} + 3 = \frac{x + 2n}{a} + \frac{a}{n}.$$

PROBLEMS

1. One sixth of a certain number, plus $\frac{1}{8}$ of the same number, equals 21. Find the number. *64*
2. The difference between $\frac{1}{3}$ of a certain number and $\frac{1}{7}$ of the same number is 6. Find the number. *42*
3. The sum of two numbers is 216. One tenth of the greater number equals $\frac{1}{8}$ of the less. Find the numbers. *144, 72*
4. The width of a rectangle is $\frac{3}{5}$ of its length. The perimeter is 200 feet. Find the area of the rectangle. *2250*
5. What number must be added to the numerator of the fraction $\frac{12}{25}$ so that the resulting fraction will be $\frac{1}{5}$ of the number added? *12*
6. Three fourths of a certain integer is $\frac{1}{3}$ the sum of the next two consecutive integers. Find the first integer. *12*
7. A certain even integer divided by 3 is 10 less than $\frac{1}{3}$ of the sum of the next two consecutive even integers. Find the first integer. *12*

8. What number added to both terms of the fraction $\frac{5}{11}$ gives a fraction whose value is $\frac{3}{5}$?

9. Separate 96 into two parts such that $\frac{1}{6}$ of their difference is 5.

10. Separate 176 into two parts such that their quotient is $\frac{2}{9}$.

11. There are two numbers whose sum is 52. If their difference is divided by their sum, the quotient is $\frac{5}{13}$. Find the numbers.

12. The weight of Mars is approximately 9 times that of the moon, and the weight of the earth is about $\frac{81}{10}$ that of the moon and Mars combined. Find the weight of the moon and of Mars in terms of the earth's weight.

13. A's age is $\frac{7}{10}$ of B's age. In 4 years A's age will be $\frac{8}{11}$ of B's age. Find their ages now.

14. At the time of her marriage a certain woman's age was $\frac{2}{3}$ that of her husband. Twelve years later her age was $\frac{3}{4}$ of his. Find their ages at the time of their marriage.

15. A is 12 years older than B. Eight years ago B was $\frac{1}{2}$ as old as A. Find their ages now.

16. The denominator of a certain fraction exceeds its numerator by 12. If 6 is added to both terms of the fraction the value of the resulting fraction is $\frac{4}{5}$. Find the fraction.

17. Jupiter has 5 more moons than Uranus, and Saturn 2 more than twice as many as Uranus; Mars has 7 fewer than Jupiter, and Neptune $\frac{1}{2}$ as many as Mars. These planets together have 26 moons. How many has each?

$$(x+5) - 7 = \text{Mars}$$

18. A triangle has the same area as a trapezoid. The altitude of the triangle is 36 feet and its base is 20 feet. The altitude of the trapezoid is $\frac{1}{3}$ that of the triangle, and one of its bases equals the base of the triangle. Find the other base of the trapezoid.

19. A marksman hears the bullet strike the target 3 seconds after the report of his rifle. If the average velocity of the bullet is 1925 feet per second and the velocity of sound is 1100 feet per second, find his distance from the target and the length of time the bullet was in the air.

20. A gunner using a modern rifle would hear the projectile strike a target 2640 yards distant $9\frac{2}{5}$ seconds after the report of the gun, provided the projectile maintained throughout its flight the same velocity it had on leaving the gun. Find this velocity if sound travels 1100 feet per second.

21. The denominator of a certain fraction is 8 more than the numerator. If the numerator is increased by 1 and the denominator decreased by 3 the value of the resulting fraction is $\frac{2}{3}$. Find the fraction.

22. The estimated total possible water power of the world is 439 million horse power. Of this North America has 8 million horse power more than South America, 17 million more than Europe, 9 million less than Asia, and 45 million more than Oceanica. Africa has 59 million horse power less than all the others combined. Find the number of horse power for each geographical division.

93. Equations containing fractions with polynomial denominators. Although no new principle is involved in the exercises of this section, they serve to review some of the most important processes of algebra.

ORAL EXERCISES

Clear the following equations of fractions, stating in each case the operation employed. Do not solve the equations.

$$1. \frac{x}{x-3} = 2.$$

Solution. Multiplying each member of the equation by $x-3$ gives $x = 2x - 6$.

$$2. \frac{x}{x-5} = 4.$$

$$9. \frac{z-6}{z+8} = \frac{3}{5}.$$

$$16. \frac{x-1}{2x} = \frac{x}{5}.$$

$$3. \frac{2x}{4-x} = 6.$$

$$10. \frac{2y+1}{1-2y} = \frac{4}{7}.$$

$$17. \frac{1}{t} = \frac{3}{2t-4}.$$

$$4. \frac{3x}{2x+1} = 5.$$

$$11. \frac{y-3}{y+3} = \frac{3}{y}.$$

$$18. \frac{5}{x} = \frac{x}{7}.$$

$$5. \frac{4x^2}{1-x^2} = 3.$$

$$12. \frac{4}{x} = \frac{5-x}{5+x}.$$

$$19. \frac{3v-1}{2v+1} = \frac{2v-1}{1+3v}.$$

$$6. \frac{z-3}{z+3} = 7.$$

$$13. \frac{3}{5x} = \frac{2x-1}{x+2}.$$

$$20. \frac{a}{x} + \frac{b}{x} = \frac{x}{a+b}.$$

$$7. \frac{z+8}{z-2} = 8.$$

$$14. \frac{x-2}{x+1} = \frac{x-1}{x+2}.$$

$$21. \frac{x}{a} + \frac{a}{x} = 2.$$

$$8. \frac{z-3}{z+5} = \frac{2}{3}.$$

$$15. \frac{x-3}{x-6} = \frac{x+6}{x+3}.$$

$$22. \frac{b}{y} + \frac{y}{b} = \frac{5}{y-b}.$$

EXAMPLE

Solve the following equation :

$$3 - \frac{2x-2}{3x+1} = 2\frac{5}{11}.$$

Solution. Both members must be multiplied by $11(3x+1)$, the L. C. M. of the denominators.

This step may be indicated as follows:

$$11(3x+1)\left[3 - \frac{2x-2}{3x+1}\right] = (2\frac{5}{11})11(3x+1).$$

Canceling where possible gives

$$33(3x+1) - 11(2x-2) = 27(3x+1);$$

whence $99x + 33 - 22x + 22 = 81x + 27.$

Transposing and collecting, $-4x = -28,$
 $x = 7.$

EXERCISES

Solve and check as directed by the teacher:

$$1. \frac{x+3}{x+1} = 2. \quad 3. \frac{x-3}{2x-1} = \frac{2}{9}. \quad 5. \frac{4}{18-2x} = \frac{1}{x}.$$

$$2. \frac{x+2}{x+4} = 3. \quad 4. \frac{3x+4}{2x-1} = 2. \quad 6. \frac{3}{x-2} + \frac{5}{x} = 2.$$

$$7. \frac{13}{x+5} + \frac{7}{2x} = 2. \quad 14. 2x - \frac{2x+7}{3-2x} = 3x.$$

$$8. \frac{x-1}{x} = \frac{6x}{3x-2}. \quad 15. \frac{x-3}{2x-5} = x - 6\frac{5}{9}.)$$

$$9. \frac{x-2}{x+6} = \frac{x}{2}. \quad (16. \frac{3y+4}{7-6y} = \frac{3y+2}{12y-1}.$$

$$10. \frac{x}{5-2x} = \frac{2x}{8x-5}. \quad 17. \frac{14y-5}{4y+5} = \frac{12-2y}{2(3+2y)}.$$

$$11. \frac{1}{3x+1} = \frac{2}{4+3x}. \quad 18. \frac{15y-4}{8} + \frac{4}{5y-4} = \frac{15y}{8}.$$

$$12. \frac{1}{2-5x} = \frac{3}{3-5x}. \quad 19. \frac{3}{5-3y} = \frac{4-3y}{3y-5} + \frac{7}{5}.$$

$$13. \frac{2x-5}{4-10x} + 12x = 0. \quad 20. \frac{7z}{z-2} = \frac{6z}{z-1} + 6.$$

$$21. \frac{9y}{y-1} = \frac{16y}{2y-1} + 1. \quad 24. \frac{z}{3z+2} - \frac{1}{7z-2} = 0.$$

$$22. \frac{3x^2 + 5x - 2}{2x^2 + 5x + 3} = \frac{3}{5}. \quad 25. \frac{4}{x^2} = \frac{3}{x+1}.$$

$$23. \frac{3}{y} + \frac{y}{3} = 2\frac{1}{6}. \quad 26. \frac{25}{x^2} = \frac{14}{3x-1}.$$

$$27. \frac{2}{2t+3} + \frac{1}{3-2t} = \frac{12t}{4t^2-9}.$$

$$28. \frac{2}{2t+1} + \frac{1}{2t-1} - \frac{8}{1-4t^2} = 0.$$

$$29. \frac{2}{x+8} = \frac{3}{5} - \frac{1}{2-x}.$$

$$30. \frac{1}{x+7} = \frac{2}{5} - \frac{1}{3-x}.$$

$$31. \frac{2t+1}{t^2+3t-4} = \frac{t+3}{t+4}.$$

$$32. \frac{t-1}{t^2-5t+6} = \frac{t-2}{2-t} - \frac{1+t}{t-3}.$$

94. Equations involving decimals. The method of solving an equation containing decimals is illustrated in the following

EXAMPLE

Solve the equation $.9x + .7 = 4.2 - .15x$.

Solution. Multiplying each member of the equation by 100,

$$90x + 70 = 420 - 15x.$$

Transposing and combining,

$$105x = 350.$$

$$x = \frac{10}{3}.$$

Check. $.9 \times \frac{10}{3} + .7 = 4.2 - .15 \times \frac{10}{3}.$

$$3 + .7 = 4.2 - .5.$$

$$3.7 = 3.7.$$

In equations containing fractions, if decimals occur in any denominator, it is often desirable to multiply both numerator and denominator of such a fraction by that power of 10 which will reduce the decimals in both the numerator and the denominator to integers. Then clear of fractions and proceed as in the foregoing example.

EXERCISES

Solve the following equations and check as directed by the teacher:

1. $.8x = 6.$

6. $3.75 = 2.15 - .25x.$

2. $.6x + .5 = .8.$

7. $.12x - 4.5 = 1.68.$

3. $.15x + 4 = .25.$

8. $.15x + .16 = .58.$

4. $.75 - .35x = .26.$

9. $.08x = .1x + 2.6.$

5. $.92 + .6x = 5.12.$

10. $.4x + 1.62 = .55x.$

11. $.3x - .8x = 16x - 49.5.$

12. $1.7x + 3.14 = -9.66 - 1.5x.$

13. $15x + 7 - 6.25x = 8.845 + 2x.$

14. $6x - 2.49x + 1.2x = 1.5 + .71x.$

15. $.36(2x + .5) - .6(1.5x - 2) = 1.2.$

16. $6(6x - 1.1) - 8.4(1.4x - 3) = 12x + .24.$

17. $.12(.5x + .05) - .15(.375x - 2) = .246.$

18. $\frac{.15x - 6.2}{4} - \frac{6.75 - .2x}{5} = \frac{3.5}{2}.$

$$19. \frac{6.2x - 1.24}{.4} - \frac{7.56 - .28x}{.35} = 7.9.$$

HINT. Multiplying numerator and denominator of each fraction on the left by 100,

$$\frac{620x - 124}{40} - \frac{756 - 28x}{35} = 7.9.$$

To avoid the possibility of repeating, in the check, a numerical error made in the solution, the check should be performed by finding the value of each fraction separately, without clearing of fractions.

$$20. \frac{.9x - .1}{1.7} = \frac{2.5 - .35x}{1.8}.$$

$$21. \frac{6.4x}{.5} + \frac{9x}{12.5} = 1.52 + 12x.$$

$$22. \frac{.15(3 - .4x)}{.16} - \frac{.25(.2x - 6)}{.8} = 2.5.$$

$$23. \frac{.3(10 - x)}{13.5} = \frac{1.5 - 5x}{48.5} + .28x - 3.8.$$

$$24. \frac{33}{3(x + 5)} + \frac{3.75}{1.5(x - 8.5)} = 0.$$

25. The formula $\frac{1}{f_1} + \frac{1}{f_2} = \frac{1}{x}$ applies to a convex lens where f_1 is the distance of an object from the lens, f_2 is the distance of the image formed by the lens, and x is a distance known as the focal length determined by the shape of the lens. If $f_1 = 3.4$ inches and $f_2 = 4.2$ inches, find x to two decimal places.

HINT. Solve the equation for x before substituting the given values.

26. If a pounds of water at a temperature t is mixed with b pounds of water at a temperature T , the temperature x of the resulting mixture (if no heat escapes) is given by the equation $at + bT = x(a + b)$. Find x if $a = 4\frac{1}{8}$ pounds, $t = 40^\circ$, $b = 3\frac{1}{2}$ pounds, and $T = 90^\circ$.

27. The velocity of sound is v in the equation $v = nl$ when n is the number of vibrations the sound makes per second and l is the length of one sound wave. The A to which an orchestra tunes makes 440 vibrations per second. If sound has a velocity of 1100 feet per second, find the wave length of the note A .

28. If a body were projected away from the earth's surface with a velocity in feet per second greater than $v = \sqrt{2gR}$ it would never return to the earth. If $g = 32$ and R equals the radius of the earth (4000 miles) expressed in feet, find this velocity.

NOTE. The introduction into Europe of the Arabic notation for numbers was one of the important events of the Middle Ages. This notation originated among the Hindus probably as early as 300 B.C. It was adopted by the Arabs, and was introduced by the Moors into Spain during the eighth century. Anyone who has tried to multiply two numbers in the Roman notation, like MDCCVII by MCXVIII, will realize the difficulties that surrounded arithmetical operations before the Arabic system was taught. Before the introduction of this system one of the principal uses of arithmetic was the determination of the day of the month on which Easter came. Roger Bacon in the thirteenth century urged the theologians "to abound in the power of numbering," so that they might carry out these computations. Business computations were made on the abacus, one form of which is a contrivance of wires and sliding balls on which arithmetical operations can be performed with great rapidity.

Though computation in the decimal system was used in Europe from the thirteenth century, the final step in perfecting the notation

was not taken until about 1600, when Sir John Napier and others made use of the decimal point in the modern sense. It was not until the beginning of the eighteenth century that it came into general use.

95. Formula for percentage. The methods of algebra may be used to advantage in dealing with many problems in percentage which are also found in arithmetic, and in solving many others which would be difficult or impossible to solve by arithmetical means alone.

ORAL EXERCISES

1. What is 5% of 60?
3. What is 4% of $x + 40$?
2. What is 5% of x ?
4. What is 7% of $4x - a$?
5. What is the interest on \$100 invested for 1 year at 6%?
6. What is the interest on \$100 invested for 6 years at 6%?
7. What is the interest on \$100 invested for t years at 4%?
8. What is the interest on P dollars invested for t years at 4%?
9. What is the interest on P dollars invested for t years at r %?
10. What is the total amount due at the end of n years if P dollars are invested at r %?

If two sums of money are x dollars and $(800 - x)$ dollars respectively, express as equations the statements made in Exercises 11-14.

11. Four per cent of the first sum equals \$20.
12. Five per cent of the first sum plus 4% of the second sum equals \$120.



John Napier

13. Six per cent of the first sum equals 5% of the second.

14. Five per cent of the first sum is \$24 more than 4% of the second.

When P dollars are invested at simple interest for t years, at a yearly rate of $r\%$, the total amount, A , accumulated is given by the following formula:

$$P + P \cdot r \cdot t = P(1 + rt) = A.$$

It should be noted carefully that the value of r is a fraction of which the denominator is 100. Thus, if the rate is 6%, the value of r is $\frac{6}{100}$, or .06.

EXAMPLE

What sum of money placed at simple interest for 3 years at 6% will amount to \$236?

Solution. Principal + interest = \$236.

Let P = the principal in dollars.

Then $.06 P \times 3$ = the interest for three years
in dollars.

Therefore $P + .18 P = 236$,

or $1.18 P = 236$;

whence $P = 200$.

Check. $200 + 200 \times .06 \times 3 = 236$.

EXERCISES

1. What sum of money placed at interest for 1 year at 6% will amount to \$318?

2. What sum of money placed at simple interest for 3 years at 5% will amount to \$368?

3. In how many years will \$475, at 4% simple interest, gain \$76?

4. In how many years will \$650, at $4\frac{1}{2}\%$ simple interest, gain \$29.25?

5. At what per cent simple interest will \$420 gain \$126 in 5 years?

Solution. $\$420 \times \text{the rate of interest} \times 5 = \$126.$

Let $r = \text{the rate of interest.}$

Then $\$420 r = \text{the interest for one year,}$

and $\$420 \cdot r \cdot 5 = \text{the interest for 5 years.}$

Therefore $\$2100 r = \$126,$

and $r = \frac{6}{100}.$

Hence the interest rate is 6%.

6. At what per cent simple interest will \$960 gain \$192 in 4 years?

7. At what per cent simple interest will \$250 amount to \$332.50 in 6 years?

8. In how many years will \$200 double itself at 5% simple interest? \$300? x dollars?

9. In how many years will \$400 treble itself at 6% simple interest?

10. A part of \$1000 is invested at 5% and the remainder at 6%. The yearly income from the two investments is \$53. Find each investment.

HINT. One part $\times .06 + \text{the other part} \times .05 = \$53.$

Let $x = \text{the number of dollars invested at 6\%.}$

Then $1000 - x = \text{the number of dollars invested at 5\%.}$

Therefore, by the conditions of the problem,

$$.06 x + .05(1000 - x) = 53.$$

11. A part of \$1500 is invested at 5% and the remainder at 4%. The total annual income from the two investments is \$66. Find the amount of each investment.

12. A sum of money at 6% interest and a second sum at 5% yield a total annual income of \$61. The first sum exceeds the second by \$100. Find each.

13. A 4% investment yields annually just as much as one at 5%. If the sum of the investments is \$2700, find each.

14. A 5% investment yields annually \$7 more than a 6% investment. If the sum of the two investments is \$800, find each.

15. A man invests part of \$4300 at 6% and the remainder at 5%. The investment at 6% yields annually \$32.50 more than the one at 5%. Find the sum invested at 5%.

16. A man invests part of \$5450 at 5% and the remainder at 6%. The yearly income from the 5% investment is \$3 more than that from the 6% investment. Find the sum invested at 6%.

17. A part of \$5000 is invested at 4% and the remainder at 5%. The total yearly income is \$220. Find the amount invested at 5%.

18. An atom of hydrogen is .063 as heavy as an atom of oxygen. Each molecule of water is made up of two atoms of hydrogen and one of oxygen. What percentage of the weight of a molecule of water is hydrogen? How many pounds of oxygen are in 100 pounds of water?

96. **Literal equations.** At this point the student should review the work on pages 124-126.

EXERCISES

Solve for x , y , z , or t and check as directed by the teacher :

$$1. 3ax - 4a^2 = 28a^2 - 5ax.$$

$$2. 39c^2 + 5cx = 2c(6x - 5c).$$

$$3. 5a^2x - 3b^2 = 2b^2 - 3a^2x - 5b^2.$$

$$4. 3(x + 5) - 6a = 9.$$

$$5. 5(x - 2) - 10c = 20.$$

$$6. 3(x - 2) + 4(2 - 3x) = 20 - 18a.$$

$$7. 5(2x - a) - 4(x - a) = \frac{18}{a} - a.$$

$$8. \frac{y}{bc} + \frac{y}{ac} = \frac{a}{bc} + \frac{1}{c}.$$

$$9. 3b(8 - 4a) = 4a(12y - 3b).$$

$$10. \frac{y}{ab} + \frac{b}{ac} = \frac{y}{ac} + \frac{1}{a}.$$

$$11. a - \frac{a^2 + 3ab}{z + a} = \frac{z^2 + a^2}{z + a} - z.$$

$$12. \frac{a^2z + 3az}{a^2 - a + 1} - 1 = \frac{z - 10a + 2}{a^2 - a + 1} + z.$$

$$13. \frac{a}{t} + \frac{3a}{2t} = \frac{5}{2}.$$

$$17. \frac{6x}{a} - \frac{6a}{x} = 5.$$

$$14. \frac{8a}{t} + \frac{24a}{t} = \frac{3}{2} + \frac{5a}{t}.$$

$$18. 10x - \frac{5b}{a} = \frac{2a}{b} - 1.$$

$$15. \frac{a^2}{x} - a = \frac{b^2}{x} + b.$$

$$19. \frac{x + c}{3} = \frac{x + 3}{c}.$$

$$16. \frac{a}{x} + \frac{x}{a} = 2.$$

$$20. \frac{x - m}{a} = \frac{x - a}{m}.$$

$$21. \frac{z - a^2}{z - c^2} = \frac{a}{c}.$$

$$22. \frac{abc + c}{ab} + \frac{c}{abz} = \frac{abc}{z}.$$

$$23. \frac{a^2}{cz} - \frac{c^2}{az} = \frac{1}{a} - \frac{1}{c}.$$

$$24. \frac{a + t}{2(c + t)} - \frac{a - t}{2c - 2t} = \frac{1}{t^2 - c^2}.$$

$$25. \frac{a - c}{3ac} - \frac{a - c}{z} = \frac{c^2}{3az} - \frac{a^2}{3cz}.$$

$$26. x - \frac{ax}{x + a} = \frac{1}{x + a}.$$

$$27. \frac{t - a}{a} - \frac{a}{t - a} = \frac{3}{2}.$$

$$30. \frac{t^2 + 1}{t} = a + \frac{1}{a}.$$

$$28. \frac{a}{2a - t} + \frac{2a - t}{a} = 2.$$

$$31. \frac{m^2}{x^2} = \frac{m + 1}{x + 1}.$$

$$29. \frac{t + 9a}{a} = \frac{t}{m - a} - \frac{m - 4a}{a - m}.$$

$$32. \frac{2r - 1}{2x - 1} = \frac{r^2}{x^2}.$$

97. Meaning of primes and subscripts. Different but related values are often represented by the same letter with smaller characters written at the right and above or below the letter used, as y' , y'' , x_0 , $4x_3$, t_m^2 , t_w . These are read *y prime*, *y second*, *x sub zero*, *4 x sub three*, *the square of t sub m*, and *t sub w* respectively. Primes and subscripts, unlike exponents, possess no numerical significance, and the student should carefully note that x_0 and x_3 are as different numerically as a and b .

The notation just explained is very convenient in physics, where L_1 and L_2 may denote different but related

lengths; W_1 and W_2 may represent two different weights; and t_0 , t_1 , and t_2 may mean three unequal but related intervals of time.

Primes are cumbersome and easily confused with exponents; hence subscripts are preferable.

The following equations are taken from algebra, geometry, and physics, where it is often necessary to express some one quantity (weight, time, distance, etc.) in terms of others.

EXERCISES

1. Solve for R ; $K = 2 \pi R H$.

(Formula for curved surface of cylinder.)

2. Solve for a ; $A = \frac{ab}{2}$.

(Formula for area of triangle.)

3. Solve for R ; $C = 2 \pi R$.

(Formula for circumference of circle. $\pi =$ approximately $\frac{22}{7}$.)

4. Solve for r and for t ; $d = rt$.

(Formula for uniform motion.)

5. Solve for a and for A ; $\frac{a}{A} = \frac{D}{360}$.

(Formula relating to the measurement of angles.)

6. Solve for C ; $\frac{D}{360} = \frac{1}{C}$.

7. Solve for r ; $I = \frac{E}{R + r}$.

(Ohm's law for a simple electrical circuit.)

8. Solve for r and for n ; $I = \frac{E}{R + nr}$.

9. Solve for r and for n ; $r = \frac{n \cdot e}{R + nr}$.

10. Find I in Exercise 9, if $n = 4$, $e = 1$, $R = 5$, and $r = 1.2$.

11. Solve for F ; $C = \frac{5}{9}(F - 32)$.

(Formula for converting thermometer readings from one scale (Fahrenheit) to another (centigrade).)

12. Solve for W_2 ; $\frac{W_1}{W_2} = \frac{L_2}{L_1}$.

13. Solve for r and for t ; $A = P(1 + rt)$.

14. Solve for P_2 ; $\frac{V_1}{V_2} = \frac{P_2}{P_1}$.

(Formula relating to volume and pressure of a gas.)

15. Solve for n and for l ; $s = \frac{n(a + l)}{2}$.

16. Find n in Exercise 15, if $a = 1$, $l = 100$, and $s = 5050$.

17. Solve for a , l , and r ; $s = \frac{rl - a}{r - 1}$.

18. Solve for a ; $\frac{D}{180} = \frac{a}{n}$.

19. Solve for t_1 ; $V_1 = V_0(1 + .00365 t_1)$.

20. Solve for b_2 ; $A = \frac{(b_1 + b_2)a}{2}$.

21. Solve for a , b , and f ; $\frac{1}{a} + \frac{1}{b} = \frac{1}{f}$.

22. Find f in the formula of Exercise 21 if $a = 10$ and $b = 18.5$.

23. The formula for the area of a circle is $A = \pi r^2$. $\pi = 3.1416$. If $r = 10$ inches, find A .

24. The formula for the volume of a cone is $V = \frac{\pi r^2 h}{3}$. Find V for a cone in which $r = 6$ inches and $h = 15$ inches.

25. Solve for W ; $E = \frac{WV^2}{2g}$.

26. In Exercise 25, if g is 32.2, $V = 40$, and $E = 10,000$, find W .

98. **The lever.** The figure given below is a diagram of a machine called a *lever*. AC is a stiff bar resting on a single support at B . This support is called the *fulcrum* and AB and BC are spoken of as *arms* of the lever.



Those who have played with a teeter board have had some experience with a lever, and they have found that, in order to balance, the heavier of two persons must sit nearer the fulcrum than the lighter one does.

In general, if the lengths of the arms of a lever are l_1 and l_2 and the corresponding weights are W_1 and W_2 , a balance results when

$$l_1 W_1 = l_2 W_2.$$

Thus, if $AB = 3$ feet and $BC = 4$ feet, a boy at A who weighs 100 pounds will balance a boy at C who weighs 75 pounds; for $3 \cdot 100 = 4 \cdot 75$.

PROBLEMS

1. A, who is 5 feet from the fulcrum, balances B, who is 7 feet from it. A weighs 102 pounds. Find the weight of B.

2. A, who weighs 100 pounds, balances B, who weighs 120 pounds. B is 6 feet 3 inches from the fulcrum. How far from it is A?

3. A, who weighs 120 pounds, balances B, who weighs 100 pounds. The distance between them is 11 feet. How far is each from the fulcrum?

4. A and B together weigh 315 pounds. They balance when A is 3 feet 9 inches from the fulcrum and B is 5 feet from it. Find the weight of each.

5. A weighs $\frac{2}{3}$ as much as B. The distance between them is 12 feet. How far from B is the fulcrum if they balance?

In each of the preceding problems the fulcrum may be thought of as the *center of gravity* of the two objects balanced.

6. The center of the moon is approximately 238,000 miles from the center of the earth. The earth is 81.5 times as heavy as the moon. Find the distance from the earth's center to the center of gravity of the earth and moon.

99. **Ratio.** The *ratio* of one number to a second number is the quotient obtained by dividing the first number by the second.

Thus the ratio of 1 to 2 is $\frac{1}{2}$; that of 7 to 3 is $\frac{7}{3}$; that of a to b is $\frac{a}{b}$.

Hence every ratio is a fraction and all fractions may be regarded as ratios. The ratio $\frac{a}{b}$ is often written $a : b$. The symbol $(:)$ may be thought of as the sign for division $(\div)$ abbreviated by omitting the bar.

EXERCISES

Simplify the following ratios by writing them as fractions and reducing the fractions to their lowest terms:

- | | | |
|-----------------------------------|---------------------------|--|
| 1. $2 : 4$. | 5. $13 : 15$. | 9. 12 feet : 12 inches. |
| 2. $3 : 12$. | 6. $14a : 21a^2$. | 10. 1 mile : 264 feet. |
| 3. $4 : 12$. | 7. $13\frac{1}{2} : 54$. | 11. $3\frac{1}{2}$ pounds : 12 ounces. |
| 4. $\frac{1}{8} : \frac{3}{24}$. | 8. $3.6 : 12$. | 12. 2.45 : 1.75. |

13. Separate 35 into two parts which are in the ratio 2 : 3.

HINT. Let one part be represented by $2x$; then the other will be $3x$.

14. Separate 24 into two parts which are in the ratio 5 : 7.

15. Separate 121 into two parts which are in the ratio of 10 to 1.

16. Separate 312 into three parts which are in the ratio of 2 to 4 to 7.

17. What number added to both terms of the ratio of 12 to 5 gives a result equal to the ratio of 2 to 1?

18. What number subtracted from both terms of the ratio of 13 to 27 gives a result equal to the ratio of 4 to 11?

19. If y is a positive number, which is the greater ratio,

$$\frac{2}{5} \text{ or } \frac{2+y}{5+y} ? \quad \frac{y}{y+3} \text{ or } \frac{2y}{2y+3} ?$$

HINT. Reduce the two fractions in each part to respectively equivalent fractions having a common denominator, and then compare the numerators.

20. From your answers to Exercise 19 state the change which occurs in the value of a proper fraction when a positive number is added to both its terms.

21. If x is a positive number, which is the greater ratio,

$$\frac{10}{7} \text{ or } \frac{10+x}{7+x} ? \quad \frac{1+x}{x} \text{ or } \frac{1+2x}{2x} ?$$

22. From your answers to Exercise 21 state the change which occurs in the value of an improper fraction when a positive number is added to both its terms.

100. Proportion. A *proportion* is a statement of equality between two ratios.

Thus $\frac{2}{4} = \frac{3}{6}$, $\frac{4}{5} = \frac{12}{15}$, and $\frac{10}{20} = \frac{1}{2}$ are proportions, for the ratios are equal in each case.

The four numbers 1, 2, 3, and 6 are said to be in proportion, for the ratio of the first pair equals the ratio of the second pair. In general, the numbers a , b , c , and d are in proportion if

$$a : b = c : d. \quad (1)$$

In (1), a and d (the first and fourth terms) are called *extremes*, and b and c are called *means*.

Since a proportion is an equality between two ratios (fractions), it is therefore an equation. Hence *any operation which may be performed on an equation may be performed on a proportion.* (See Axioms, pp. 56-58.)

Thus, in the proportion $\frac{a}{b} = \frac{c}{d}$ both members may be multiplied by bd , giving $ad = bc$. Here the first member is the product of the extremes of the proportion, and the second member is the product of the means.

Therefore, *in any proportion the product of the extremes equals the product of the means.*

EXERCISES

Form proportions from the following by supplying the missing terms :

$$1. \frac{2}{3} = \frac{12}{?}$$

$$4. \frac{36}{40} = \frac{9}{?}$$

$$7. \frac{5}{7} = \frac{?}{35}$$

$$2. \frac{1}{2} = \frac{4}{?}$$

$$5. \frac{15}{?} = \frac{3}{4}$$

$$8. \frac{2}{9} = \frac{?}{81}$$

$$3. \frac{12}{10} = \frac{?}{5}$$

$$6. \frac{13}{52} = \frac{1}{?}$$

$$9. \frac{12}{3} = \frac{4}{?}$$

Solve the following for x :

$$10. 12 : 7 = x^2 : 21. \quad 14. m : 1 = x : n. \quad 18. \frac{6}{x} = \frac{5x + 9}{12}.$$

$$11. \frac{3x + 1}{2x + 1} = \frac{7}{5}. \quad 15. \frac{1}{x} = \frac{x}{25}. \quad 19. r : x = r : s.$$

$$12. 3 : \frac{x}{2} = \frac{x}{2} : 12. \quad 16. \frac{4}{3} = \frac{12}{x^2}. \quad 20. \frac{2x}{3} : p = \frac{p}{2} : q.$$

$$13. 5 : 1 = 1 : \frac{1}{x}. \quad 17. x : a = b : c. \quad 21. x : 1 = 1 : v.$$

101. Variation. If the price of a certain kind of cloth is 50 cents a yard, the number of cents, A , that n yards would cost is given by the formula $A = 50n$. In this case the ratio $A : n = 50$. If n_1 and n_2 denote the number of yards of cloth bought for A_1 and A_2 cents respectively, then $\frac{A_1}{n_1} = \frac{A_2}{n_2} = 50$. In other words, the amount of money paid for the cloth is proportional to the number of yards of the cloth purchased.

Another way of expressing this idea is to state that the total cost of the cloth *varies directly* as the number of yards of cloth purchased.

When one variable varies directly as another, the ratio of any corresponding pair of values of the variables is constant.

The volume of gas in a tank varies as the temperature. That is, if V_1 and V_2 are the volumes corresponding to the temperatures t_1 and t_2 , then $\frac{V_1}{t_1} = \frac{V_2}{t_2}$. Furthermore, if V is the volume at *any* temperature t , and V_1 is the volume at a particular temperature t_1 , then $\frac{V}{t} = \frac{V_1}{t_1}$, or $V = \left(\frac{V_1}{t_1}\right)t$.

Since V_1 and t_1 are definite numbers, $\frac{V_1}{t_1}$ is a constant. This illustrates the fact that *if V varies as t , then $V = kt$.*

That is, the statement that any number, as x , varies as another number, as y , is equivalent to the equation $x = ky$, where k is a constant. If the value of this constant, and a particular value of either x or y , are known, then the value of the other variable can be found.

In the first illustration above, $A = 50n$. That is, $k = 50$. If A , the amount of money expended, is known, then the number of yards of cloth can be found.

EXAMPLES

1. A train travels at an average rate of 40 miles per hour. Express this as a formula, and tell what variables vary directly as each other.

Solution. Let d denote the number of miles traveled by the train in t hours.

Then
$$d = 40t, \text{ or } \frac{d}{t} = 40.$$

Here the number of miles traveled varies directly as the time.

2. The ratio of the area of a circle to the square of its radius is constant and equal to the number π ($= 3.14 +$). Express this as a formula and tell what variables vary directly as each other.

Solution. Let A denote the area of any circle and r its radius.

Then
$$A = \pi r^2, \text{ or } \frac{A}{r^2} = \pi.$$

The area of a circle varies directly as the square of its radius.

EXERCISES

In the following exercises determine which are true proportions:

1. $3 : 7 = 6 : 15$.

3. $25 : 3 = 75 : 10$.

HINT. Express the ratios in fractional form.

4. $2 : 3 = 14 : 20$.

5. $3 : 4 = 36 : 48$.

2. $3 : 5 = 24 : 40$.

6. $15 : 9 = 5 : 3$.

7. Two workmen are paid "piecework rates"; that is, they receive a certain sum of money for each article they make. If the rate is 2 cents per piece, state a formula showing what their earnings will be for n articles. If one man produces 150 pieces and the other produces 130 pieces an hour, what will their hourly earnings be?

8. A certain map is drawn so that it represents actual distances in the proportion of 1 inch to 1 mile. How long would a lake 5 miles long appear on the map? A river 500 feet wide would have what width on the map?

9. The cost of linoleum is proportional to the area bought. If a piece 4 feet by 12 feet cost \$12, what was the rate per square foot? How much would 100 square feet cost?

10. A man invests \$5000 and receives \$200 annual interest. How much interest will he receive from \$4500? How much must he invest to receive \$350 per year at the same interest rate? What was the rate of interest?

11. The distance (S), in miles, traveled by a train moving uniformly is expressed by the formula $S = VT$, in which V is the rate of the train in miles per hour, and T is the time in hours during which the train is moving. If the train travels 100 miles in $2\frac{1}{2}$ hours, how far can it go in 4 hours? How long will it take it to cover 125 miles? What is its speed?

12. The cost of a building is often expressed in terms of its cost per cubic foot of capacity. If the rate is 50 cents per cubic foot, how much will a building 100 feet long, 50 feet wide, and 40 feet high cost? A building 75 feet by 75 feet by 100 feet cost \$2502,500 to construct. What was the rate per cubic foot?

Write each of the following rules as a direct variation, using letters to represent the quantities involved:

13. Interest varies directly as time.
14. The area of a rectangle varies directly as the product of the length times the width.
15. The interest on a sum of money varies directly as the interest rate.
16. The speed with which a body is falling at any instant varies directly as the length of time during which it has been falling.
17. The usual cost of a railway ticket varies directly as the distance to be traveled.
18. The weight of any given size of wire is proportional to the length of the wire. A certain size of wire has 100 feet to the pound. How much wire is there in a roll weighing 75 pounds?
19. The volume of a sphere is proportional to the cube of the radius. What is the ratio between the volumes of two spheres one of which has a diameter $\frac{1}{2}$ as great as the other?
20. The area of a circle is proportional to the square of the radius. What is the area of a circle of 2-foot radius if a circle of 1-foot radius has an area of 3.14 square feet?

21. A man pays 4 cents a mile for a railway ticket, and the total cost is \$8. How far is he traveling? Express the relation by means of a formula.

22. The area of the surface of a cube is proportional to the square of any edge. What is the edge of a cube 6 square feet in area if a cube with an edge of 3 feet has an area of 54 square feet?

REVIEW PROBLEMS

1. Separate 318 into two parts such that their quotient is 7.

2. Separate 170 into two parts such that $\frac{2}{3}$ of the greater shall equal $\frac{3}{4}$ of the less.

3. Separate $\frac{5}{3}$ into two parts such that $\frac{1}{3}$ of one part shall equal $\frac{1}{5}$ of the other.

4. Find two numbers whose sum is 162 and such that the greater divided by the less gives a partial quotient of 3 and a remainder of 14.

$$\text{HINT.} \quad \frac{\text{Dividend}}{\text{Divisor}} = \text{Partial Quotient} + \frac{\text{Remainder}}{\text{Divisor}}.$$

Let $x =$ the less number.

Then $162 - x =$ the greater number.

$$\text{That is,} \quad \frac{162 - x}{x} = 3 + \frac{14}{x}.$$

Solve and check as usual.

5. Separate 149 into two parts such that one divided by the other gives a partial quotient of 4 and a remainder of 4.

6. The sum of two numbers is 1516. The greater divided by the less gives a partial quotient of 5 and a remainder of 130. Find the numbers.

7. Separate $\frac{11}{5}$ into two parts such that their product is greater by $\frac{1}{25}$ than the square of the smaller part.

8. The sum of two numbers is 36. Four times the greater number exceeds 50 by twice as much as 36 exceeds the less. Find the numbers.

9. A boy's age now is $\frac{3}{5}$ of what it will be 8 years hence. How old is he now?

10. One sixth of a certain man's age 8 years ago equals $\frac{1}{10}$ of his age 8 years hence. What is his age now?

11. A collection of nickels and quarters contains 60 coins. Their total value is \$11.60. How many are there of each?

12. Twenty-three coins, dimes and quarters, have the value \$4.70. How many are there of each?

13. The square of half a certain even number is 92 less than $\frac{1}{4}$ the product of the next two consecutive even numbers. Find the numbers.

14. A rectangle is four times as long as it is wide. If it were 8 feet shorter and 3 feet wider, its area would be 104 square feet more. Find its length and breadth.

15. A rectangle is $\frac{2}{3}$ as broad as it is long. If its length were doubled and its breadth diminished by 32, its area would be 1584 square feet. What are its dimensions?

16. It costs as much to sod a square piece of ground at 20 cents per square yard as to fence it at 15 cents per running foot. Find the side of the square.

17. A rectangular court is twice as long as it is wide. It costs half as much to fence it at $66\frac{2}{3}$ cents per running foot as to seed it at 5 cents per square yard. Find its dimensions.

18. A rectangular picture twice as long as it is wide is surrounded by a frame 3 inches wide. The area of the frame is 288 square inches. Find the dimensions of the picture.

19. A man bought apples at 16 cents per dozen. He sold $\frac{1}{4}$ of them at the rate of 3 for 7 cents, but the rest were not so good, and he had to sell them at the rate of 2 for 3 cents. He made a profit of \$1.26 on the entire transaction. How many dozen apples did he buy?

20. A baseball team has won 30 games and lost 24. It has 16 games yet to play. How many of these may it lose and yet win 60 per cent of the total?

21. A student has an average grade of 86 for four subjects. What grade must she make in a fifth subject so that her average will be 88?

22. A can do a piece of work in 3 days, B in 4 days, and C in 5 days. How long will it take them, working together, to do the same work?

Solution. By the conditions of the problem A does $\frac{1}{3}$ of the work in one day, B does $\frac{1}{4}$ of the work in one day, and C does $\frac{1}{5}$ of the work in one day. Let x represent the number of days required by A, B, and C together to do the work.

Then let $\frac{1}{x}$ = the fractional part of the work the three together do in one day.

Therefore
$$\frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{1}{x}, \text{ etc.}$$

Solving,
$$x = \frac{60}{47}.$$

Check as usual.

23. A can do a piece of work in 6 days and B in 5 days. How many days will they require, working together?

24. A can do a piece of work in 6 days, B and C each in 8 days, D in 15 days. How many days will they require, working together.

25. A can do a piece of work in 6 days, B in $4\frac{1}{2}$ days. How many days will they require, working together?

26. A can do a piece of work in 9 days, A and B, working together, in $4\frac{4}{5}$ days. How long would it take B alone?

27. A can do a piece of work in $5\frac{1}{4}$ days, B in $4\frac{1}{5}$ days, A, B, and C together in $1\frac{2}{5}$ days. How long would it take C alone?

28. A can do a piece of work in 8 days. After he has worked 3 days, B joins him and they finish the work in 2 more days. How long would it have taken B to do the work alone?

HINT. What fractional part of the work does A do in 1 day? in 3 days? in 5 days? What fractional part does B do in 1 day? in 2 days?

29. A can do a piece of work in 4 days, B in 6 days. After A has worked alone for 1 day, B joins him and they finish the job together. How long does it take them to complete the work?

30. A mass of lead and tin weighing 64 pounds contains 12 pounds of tin. How many pounds of tin must be added so that 50 pounds of the mixture will contain 15 pounds of tin?

31. Two motorists start at the same time to ride from A to B, 240 miles distant. One travels 10 miles an hour more than the other. The faster motorist reaches B and at once starts back, meeting the slower one at C, 192 miles from A. Find the rate of each.

Solution. The problem states that the two travel at different rates, that they travel different distances, but that the time is the same for each. Hence the equation must be formed by expressing the time t , or $\frac{d}{r}$, for each and equating the two expressions for t .



The two men together cover twice the distance from A to B, or 480 miles. As the slower one travels 192 miles, the faster travels $480 - 192$, or 288 miles. If x equals the rate of the slower motorist in miles per hour, we have:

	d IN MILES	r IN MILES PER HOUR	$\frac{d}{r} = t$ IN HOURS
Slower motorist	192	x	$\frac{192}{x}$
Faster motorist	$480 - 192 = 288$	$x + 10$	$\frac{288}{x + 10}$

Hence
$$\frac{192}{x} = \frac{288}{x + 10}.$$

Solving, we obtain $x = 20$, the rate of the slower motorist in miles per hour, and $x + 10 = 30$, the rate of the faster motorist.

Check. $\frac{192}{20} = 9.6$ and $\frac{288}{30} = 9.6$.

32. Two motorists, A and B, start at the same time to ride from X to Y, 180 miles distant. A travels 8 miles per hour less than B. The latter reaches Y and at once turns back, meeting A 20 miles from Y. Find the rate of each.

33. A train runs 280 miles. On the return trip it increases its rate by 5 miles an hour and makes the run in an hour less time. Find the rates going and returning.

34. An automobile makes a run of 120 miles. The chauffeur then increases the speed by 10 miles an hour and returns over the same route in 2 hours less time. Find the rates, going and returning.

35. A bicyclist traveling 18 miles per hour was overtaken $11\frac{1}{3}$ hours after he started by an automobile which left the same starting point 3 hours and 20 minutes later. What was the rate of the automobile?

36. An automobile makes a run of 300 miles. On the return trip the chauffeur decreases the speed by 5 miles an hour and requires 5 hours longer to cover the distance. Find the speed each way.

37. A man travels at a uniform rate from A to B, 150 miles distant. He travels the first 90 miles without stopping. The rest of the journey, including a delay of 3 hours, takes the same time as the first part. Find his speed.

HINTS. By reading the problem we discover that the distances covered in the first and second portions of the journey are different, that the time of travel is not the same for each, but that the rate throughout is the same. Hence one should find the two expressions for the rate r , or $\frac{d}{t}$, and set them equal to each other.

38. A leaves a certain point and walks at the rate of $3\frac{1}{2}$ miles per hour. Two and a half hours later B leaves the same point and drives in the opposite direction at the rate of 12 miles per hour. How much time must elapse after A starts before they will be 100 miles apart?

39. A and B start at the same time from two points 360 miles apart and travel toward each other. A's rate is 3 miles per hour less than B's. The latter, having been delayed 2 hours on the way, has traveled the same distance as A when they meet. Find the speed of each.

40. A man rows 4 miles per hour in still water. He finds that it requires 5 hours to row upstream a certain distance and 3 hours to return. Find the speed of the current.

HINT. Let x = the speed of the current. Then $4 - x$ = the speed of the boat upstream, and $4 + x$ = the speed downstream.

41. A man who can row $4\frac{1}{2}$ miles per hour in still water rows up a stream the rate of whose current is 2 miles per hour. After rowing back he finds that the entire trip took 8 hours. How far upstream did he go?

42. A man who can row $3\frac{3}{4}$ miles per hour in still water rows downstream and returns. The rate of the current is $1\frac{1}{4}$ miles per hour, and the time required for the round trip is 9 hours. How many hours did he take to return?

43. The first transatlantic non-stop flight was made on June 14-15, 1919, by Alcock and Brown from St. John's, Newfoundland, to Clifden, Ireland, a distance of 1960 miles. Had their speed been 21 miles per hour less, the time would have been 3.4 hours more. State the equation which expresses this condition.

44. A mixture contains 3 gallons of gasoline and 5 gallons of kerosene. How many gallons of kerosene must be added to make a mixture that is $\frac{3}{4}$ kerosene?

Solution. In the final mixture,

$$\frac{\text{kerosene}}{\text{kerosene} + \text{gasoline}} = \frac{3}{4}.$$

Let x = gallons of kerosene to be added.

Then $5 + x$ = gallons of kerosene in final mixture,

and $8 + x$ = total gallons of final mixture.

But

$$\frac{5+x}{8+x} = \frac{3}{4}.$$

$$20 + 4x = 24 + 3x,$$

and

$$x = 4.$$

Check. Total gallons of final mixture = $5 + 3 + 4 = 12$ gallons.

Kerosene in final mixture = $5 + 4 = 9$ gallons.

$$\frac{9}{12} = \frac{3}{4}.$$

45. A cook wishes to make salad dressing consisting of $\frac{2}{3}$ olive oil and $\frac{1}{3}$ vinegar. She has a mixture of 1 cup of oil and 1 cup of vinegar. How much oil must she add to produce the required mixture?

46. How much milk must be added to a 5-quart mixture of equal parts of milk and water to give a resulting mixture which is $\frac{2}{3}$ milk?

47. A man has 5 gallons of gasoline and an unlimited supply of a mixture which is $\frac{1}{4}$ gasoline and $\frac{3}{4}$ kerosene. How much of this mixture must he add to his gasoline to produce a mixture which is half gasoline and half kerosene?

48. A painter wishes to make a mixture consisting of $\frac{1}{16}$ linseed oil and $\frac{15}{16}$ paint. He accidentally pours 1 quart of oil into a gallon of paint. How much paint must he add to produce a mixture in the proportion which he wishes?

CHAPTER XVIII

LINEAR SYSTEMS

102. Definitions. A simple or *linear* equation in one or more unknowns is one which may be put in such form that

- (a) no unknown appears in any denominator;
- (b) only one unknown appears in any term;
- (c) only the first power of any unknown is involved.

The following equations are linear: $5x - 2y = 3$; $2m + 3r - 2t = 0$. The following equations are not linear: $2x + 5xy - 7y = 2$; $\frac{4}{u} - \frac{6}{v} + \frac{2}{w} = 2$; $x^2 + 2x + 5y - 3 = (2x - 1)(5y + 8)$.

In a previous chapter we have seen that a simple equation in one unknown has only one root. In other words, the value of the unknown in such an equation is a constant.

Thus the value of the unknown in the equation $5x + 3 = 8$ is the number 1, and this is the only root of the equation.

A linear equation in two unknowns is satisfied by an unlimited number of pairs of values of the two unknowns. A single equation in more than one unknown is called an *indeterminate* equation. The unknowns are sometimes called variables.

Thus, the equation $x + y = 7$ is satisfied by any pair of numbers whose sum is 7. Evidently, if one includes positive, negative, and fractional numbers, there is no limit to the number of pairs whose sum is 7. (See page 276.)

Two or more equations involving two or more unknowns are called a *system* of equations.

A system of two equations both of which are satisfied by the same values of the unknowns is called a *simultaneous system* of equations.

In order that the two equations $x + y = 9$ and $x - y = 1$ may form a simultaneous system, the two numbers that satisfy both equations must be such that their sum is 9 while their difference is 1. These conditions are satisfied by $x = 5$, $y = 4$.

A *set* of values (one for each unknown) which satisfies an equation in two or more unknowns is sometimes called a solution of the equation; and a set which satisfies a system is often called a solution of the system. In this book, however, the word *solution* will be used to denote the *process of solving* either a single equation or a system. The values of the unknown which satisfy an equation in one unknown will be called *roots*, and a set of values for the unknowns satisfying an equation in two or more unknowns, or a system of such equations, will be called a *set of roots*.

ORAL EXERCISES

In Exercises 1-3 find the value of x corresponding to each of the values for y indicated at the right, and in Exercises 4-6 the values of y corresponding to the given values of x :

1. $x + y = 0$. $y = 0, 1, 3$. 4. $2x - y = 4$. $x = -3, 6, 5$.
2. $x + 2y = 3$. $y = -1, 3, 6$. 5. $x + 6y = 5$. $x = 4, 2, -1$.
3. $x - y = 1$. $y = 2, -5, 0$. 6. $3x - 2y = 6$. $x = 3, -9, 7$.

In Exercises 7-12 determine which of the pairs of numbers written at the right of each equation satisfies that

equation. (The first number of a pair always denotes the value of x and the second number the value of y .)

7. $5x - 2y = 3$. (7, 6); (1, 1); (4, 2).
8. $x + y = 4$. (2, 5); (6, 8); (1, 3).
9. $2x - 5y = 2$. (5, 2); (6, 2); (8, 1).
10. $x + 2y = 0$. (1, 2); (5, -1); (4, -2).
11. $2x - y = 3$. (2, 1); (7, 6); (3, 2).
12. $2x + 4y = 8$. (3, 4); (2, 1); (6, -1).

In Exercises 13-16 find two pairs of numbers which satisfy each equation, and two other pairs which do not.

13. $3x + 4y = 6$.
14. $2x - 5y = 1$.
15. $3x - y = 0$.
16. $x + 9y = -3$.

103. Solution by addition and subtraction. It was shown in the previous section that there is an unlimited number of sets of roots of a given linear equation in x and y , and that there is also an unlimited number of pairs of values of x and y which do not satisfy the equation. We now proceed to give an algebraic method of determining whether or not there is any common set of roots for two given linear equations. It turns out that there is usually one and only one such set of roots. This set may always be found by the method illustrated in the following

EXAMPLES

1. Solve the system $\begin{cases} 2x + y = 4, & (1) \\ 2x - 3y = -4. & (2) \end{cases}$

Solution. Eliminate x first, thus:

$$(1) - (2), \quad 4y = 8. \quad (3)$$

$$(3) \div 4, \quad y = 2. \quad (4)$$

Substituting 2 for y in either (1) or (2), say in (1),

$$2x + 2 = 4. \quad (5)$$

Solving (5), $x = 1. \quad (6)$

Check. Substituting 1 for x and 2 for y in (1) and (2) gives the identities

$$2 + 2 = 4$$

and $2 - 6 = -4.$

2. Solve the system $\begin{cases} 4x + 3y = 0, & (1) \\ 10x - 7y = 58. & (2) \end{cases}$

Solution. Eliminate x first, thus:

$$(1) \cdot 5, \quad 20x + 15y = 0. \quad (3)$$

$$(2) \cdot 2, \quad 20x - 14y = 116. \quad (4)$$

$$(3) - (4), \quad 29y = -116. \quad (5)$$

$$(5) \div 29, \quad y = -4. \quad (6)$$

Substituting -4 for y in (2),

$$10x + 28 = 58. \quad (7)$$

Solving (7), $x = 3. \quad (8)$

Check. Substituting 3 for x and -4 for y in (1) and (2) gives

$$12 - 12 = 0$$

and $30 + 28 = 58.$

Either x or y could have been eliminated first. The multipliers necessary to eliminate y are 7 and 3; the multipliers necessary to eliminate x are 5 and 2.

Since the object of solving a system of equations in x and y is the discovery of a set of values for x and y which will satisfy both equations at the same time, we are therefore safe in saying that x denotes the same number in both equations (1) and (2). The fact that the x 's disappear when equation (2) is subtracted from equation (1) in Example 1 does not depend on the fact that the x 's look

alike, but solely upon the fact that the value represented by x has been eliminated from the first members of (1) and (2), hence a resulting equation is found in which x does not appear. The line of reasoning presented above is equally true for the unknown y .

When the notation (3) — (4) is used in a solution, as illustrated in Example 2, page 255, it indicates the subtraction of the first member of equation (4) from the first member of equation (3), the subtraction of the second member of equation (4) from the second member of equation (3), and the writing of the two results as an equation. The process of adding the corresponding members of the two equations is indicated by writing (3) + (4).

The notation (1) $\cdot$ 5 indicates that both members of equation (1) are multiplied by 5, and (5) $\div$ 29 indicates that both members of (5) are divided by 29.

With the meanings just explained it is customary to speak of the addition or the subtraction of two equations and of the multiplication or division of an equation by a number.

The preceding method of solving a system of equations is summarized in the

RULE. *If necessary, multiply the first equation by a number, and the second equation by another number, such that the coefficients of the same unknown in each of the resulting equations will be numerically equal.*

If these coefficients have like signs, subtract one equation from the other; if they have unlike signs, add. Then solve the equation thus obtained.

Substitute the value just found, in the simplest of the preceding equations which contain both unknowns, and solve for the other unknown.

CHECK. *Substitute for each unknown in the original equations its value as found by the rule. If the resulting equations are not obvious identities, simplify them until they become such.*

An attempt to solve by the rule the pair

$$\begin{cases} 3x - 6y = 40, & (1) \\ x - 2y = 8, & (2) \end{cases}$$

gives $3x - 6y = 40,$ (3)

$$3x - 6y = 24. \quad (4)$$

$$(3) - (4), \quad 0 = 16, \text{ which is false.}$$

This result indicates that (1) and (2) do not form a simultaneous system but are *incompatible* equations.

An attempt to solve by the rule the system $\begin{cases} x + 2y = 8, \\ 3x + 6y = 24, \end{cases}$ gives $0 = 0$. Here each member of the second equation is just three times the corresponding member of the first equation. If we choose to regard the two equations as really different, which is not at all necessary, we say that they have an infinite (unlimited) number of sets of roots. Two or more equations having this property constitute an *indeterminate system*.

EXERCISES

Solve the following systems and check the sets of roots obtained:

1. $\begin{cases} x + y = 6, \\ x - y = 2. \end{cases}$

2. $\begin{cases} x + y = -1, \\ x - y = 5. \end{cases}$

3. $\begin{cases} x + 2y = 7, \\ x - y = -2. \end{cases}$

4. $\begin{cases} x + y = 8, \\ x + 3y = 12. \end{cases}$

5. $\begin{cases} x - y = 1, \\ 2x + y = 14. \end{cases}$

6. $\begin{cases} 3x + 2y = 38, \\ 2x - y = 23. \end{cases}$

7. $\begin{cases} x + 5y = 9, \\ 2x + 4y = 6. \end{cases}$

8. $\begin{cases} 2y + x = -3, \\ x + y = -1. \end{cases}$

9. $\begin{cases} 3x + 5y = -5, \\ x + y = -1. \end{cases}$

10. $\begin{cases} 5x + 4y = 10, \\ 6x - 3y = -27. \end{cases}$

$$\left(\begin{array}{l} 11. \quad 3x + 2y = -12, \\ \quad \quad 2x + 5y = -19. \end{array} \right.$$

$$12. \quad \begin{array}{l} 3x - y = -1, \\ 12x + 11y = 56. \end{array}$$

$$13. \quad \begin{array}{l} 9m + 5p = 109, \\ 7m - 10p = 57. \end{array}$$

$$14. \quad \begin{array}{l} 3r - 2s = 12, \\ 4r + 6s = 68. \end{array}$$

$$15. \quad \begin{array}{l} 6p + 5q = 60, \\ 12p + 13q = 138. \end{array}$$

$$\left(16. \quad \begin{array}{l} t + 10u = 57, \\ 3t - 5u = -39 \end{array} \right)$$

$$\left(17. \quad \begin{array}{l} 10x + 7y = 27, \\ 4x - 9y = -1. \end{array} \right.$$

$$18. \quad \begin{array}{l} 13x + 7y = -79, \\ 12x - 25y = -10. \end{array}$$

$$19. \quad \begin{array}{l} 5x - 7y = 78, \\ 3x + 13y = -108. \end{array}$$

$$\left(20. \quad \begin{array}{l} 2x - 5z = -27, \\ 7x + 13z = -3 \end{array} \right)$$

$$\left(21. \quad \begin{array}{l} 13x - 5y = -14, \\ 5x + 3y = 34. \end{array} \right)$$

$$\left(22. \quad \begin{array}{l} 2m + 5n = 15, \\ m - 4n = 1. \end{array} \right)$$

104. Solution by substitution. The method of solving a system of two linear equations by substitution is illustrated in the following

EXAMPLE

$$\text{Solve the system } \begin{cases} x + 5y = -11, & (1) \\ 2x - 3y = 17. & (2) \end{cases}$$

$$\text{Solution. From (1),} \quad x = -5y - 11. \quad (3)$$

Substituting $-5y - 11$ for x in (2),

$$2(-5y - 11) - 3y = 17. \quad (4)$$

$$\text{Simplifying, } -10y - 22 - 3y = 17. \quad (5)$$

$$\text{Collecting,} \quad -13y = 39, \quad (6)$$

$$\text{or,} \quad y = -3. \quad (7)$$

Substituting -3 for y in (3),

$$x = 15 - 11 = 4. \quad (8)$$

Check. Substituting 4 for x and -3 for y in (1) and (2) gives

$$4 - 15 = -11$$

and

$$8 + 9 = 17.$$

The method of the preceding solution for solving a system of two linear equations is stated in the

RULE. *Solve either equation for one unknown in terms of the other.*

Substitute this value in place of the unknown in the equation from which it was not obtained, and solve the resulting equation.

Substitute the definite value just found in the simplest of the preceding equations which contain both unknowns, and solve, thus obtaining a definite value for the other unknown.

CHECK. *See page 256.*

The method of substitution emphasizes the fact that the values of x and y which are sought are the same in both equations. Hence an expression for an unknown obtained from one equation is substituted for that unknown in the other equation. This method is useful when one of the unknowns can be expressed in terms of the other without fractions or when simple fractions only are involved.

EXERCISES

Solve by the method of substitution and check the sets of roots obtained:

1. $m - 2n = 1,$
 $2m - n = 5.$
2. $x + 3y = 14,$
 $3x - y = 2.$
3. $9s + 10t = 113,$
 $4s - 3t = 13.$
4. $y + z = 1,$
 $3y - 10z = -36.$
5. $3x + 5y = 61,$
 $10x - y = 62.$
6. $3s + 2t = 7,$
 $s + t = -1.$
7. $x + 11y = -42,$
 $x - 7y = 30.$
8. $5x = 2y + 3,$
 $3y - 6x = -3.$
9. $2x - 7y = 37,$
 $2x + y = -3.$
10. $43w + 10s = 189,$
 $5w - 3s = -3.$
11. $2x - 19y = -212,$
 $3y + 7x = 92.$
12. $3x + 2y = 1,$
 $5x - 4y = 3.$

105. Simultaneous equations containing fractions. The method of solving a system of two linear equations containing fractions is given in the following

EXAMPLE

$$\text{Solve the system } \begin{cases} \frac{2x}{5} + \frac{3y}{2} = \frac{34}{5}, & (1) \\ \frac{x}{3} + \frac{3y}{7} = \frac{50}{21}. & (2) \end{cases}$$

$$\begin{array}{ll} \text{HINTS. } (1) \cdot 10, & 4x + 15y = 68. & (3) \\ (2) \cdot 21, & 7x + 9y = 50. \end{array}$$

The system (3) and (4) can now be solved either by addition and subtraction or by substitution.

As in the foregoing solution, it is usually best to clear the equations of fractions and write them in the form of (3) and (4) before attempting to eliminate one of the unknowns. Equations (3) and (4) are each in what is known as the *general form* of a linear equation in two unknowns. This form is represented for all such equations by $ax + by = c$, where a , b , and c denote numbers, or known literal expressions.

EXERCISES

Solve the following systems and check the sets of roots obtained :

$$(1.) \begin{cases} 2x + \frac{3y}{2} = \frac{39}{2}, \\ x - \frac{y}{2} = \frac{27}{2}. \end{cases}$$

$$(2.) \begin{cases} \frac{3x}{7} - 2y = -\frac{47}{7}, \\ \frac{5y}{4} + \frac{3x}{2} = \frac{19}{2}. \end{cases}$$

$$3. \begin{cases} \frac{x+y}{7} + \frac{x-y}{5} = \frac{18}{35}, \\ \frac{x+y}{3} - \frac{x-y}{4} = \frac{23}{12}. \end{cases}$$

$$(4.) \begin{cases} \frac{2x}{5} + \frac{y}{4} = \frac{23}{20}, \\ \frac{9x}{4} + y = \frac{21}{4}. \end{cases}$$

$$(5.) \begin{cases} \frac{x+y}{3} + \frac{x-y}{5} = \frac{6}{5}, \\ \frac{2x-2y}{7} - \frac{3x-3y}{4} = -\frac{13}{28}. \end{cases}$$

$$6. \begin{cases} \frac{4x+3y}{6} - \frac{2x+5y}{2} = \frac{35}{6}, \\ \frac{x+y}{2} + \frac{x-y}{3} = -\frac{1}{12}. \end{cases}$$

$$7. \begin{cases} \frac{m+2n}{5} - \frac{2n}{3} = \frac{11}{15}, \\ \frac{4m-n}{2} + m = 4. \end{cases}$$

$$11. \begin{cases} x + \frac{5y}{3} = \frac{5}{4}, \\ \frac{7x}{3} - \frac{2y}{9} = -\frac{43}{36}. \end{cases}$$

$$8. \begin{cases} \frac{x+y}{1} + \frac{2x-2y}{3} = \frac{4}{3}, \\ \frac{x-y}{3} - \frac{x+y}{7} = \frac{2}{3}. \end{cases}$$

$$12. \begin{cases} \frac{2a-5b}{3} + \frac{2a}{5} = \frac{22}{45}, \\ \frac{4a+2b-3}{13} + a = \frac{11}{6}. \end{cases}$$

$$9. \begin{cases} \frac{r+s}{2} + \frac{r-s}{3} = \frac{2}{3}, \\ \frac{2r-3s}{5} + 2r = 3. \end{cases}$$

$$13. \begin{cases} \frac{1}{x} + \frac{1}{y} = \frac{6}{5}, \\ \frac{1}{x} - \frac{1}{y} = -\frac{4}{5}. \end{cases}$$

~~$\frac{2}{x} = \frac{2}{5}$~~
 $x=5$
 $y=5$

$$10. \begin{cases} \frac{l+m-2}{3} + \frac{m}{2} = -\frac{10}{3}, \\ \frac{2m-l+3}{2} - \frac{5m+1}{3} = \frac{1}{2}. \end{cases}$$

HINT. Solve without clearing of fractions, using addition and subtraction.

~~14.~~
$$\begin{cases} \frac{2}{x} + \frac{2}{y} = 3, \\ \frac{3}{x} - \frac{5}{y} = \frac{1}{2}. \end{cases}$$

$$15. \begin{cases} \frac{2}{m} + \frac{3}{n} = 1, \\ \frac{4}{n} - \frac{5}{m} = -\frac{19}{3}. \end{cases}$$

$$16. \begin{cases} 7r + \frac{3}{t} = -6, \\ 2r + \frac{5}{t} = -\frac{1}{3}. \end{cases}$$

$$17. \begin{cases} \frac{5}{K} + 2L = \frac{7}{2}, \\ \frac{6}{K} - 5L = -18. \end{cases}$$

$$18. \begin{cases} \frac{x+y-5}{2} + \frac{x-y}{3} = -\frac{1}{3}, \\ \frac{x-y+2}{6} - \frac{x+y+1}{3} = \frac{1}{2}. \end{cases}$$

$$19. \begin{cases} \frac{x+z-1}{5} + z - 2 = \frac{21}{5}, \\ \frac{2x-z}{3} + \frac{z+x}{4} = \frac{17}{12}. \end{cases}$$

$$20. \begin{cases} \frac{5}{x+y} + \frac{7}{3x-5y+4} = -2, \\ x - \frac{5y}{3} = \frac{10}{3}. \end{cases}$$

$$21. \begin{cases} \frac{9x-y}{7x+2y} = \frac{3}{4}, \\ 2x-5y = -22. \end{cases}$$

$$22. \begin{cases} \frac{5x}{2x-5y+3} = -1, \\ \frac{x}{2} + \frac{y}{1} = \frac{3}{2}. \end{cases}$$

$$23. \begin{cases} \frac{2x-4y}{3} = \frac{9x+10y+72}{5}, \\ 2x - \frac{5y}{6} = \frac{49}{6}. \end{cases}$$

$$24. \begin{cases} \frac{4x-3y+2}{5} = 2y + \frac{49}{5}, \\ \frac{2x}{3} + \frac{5y}{2} = -\frac{37}{6}. \end{cases}$$

$$25. \begin{cases} \frac{3x}{2} + \frac{y}{3} = \frac{11}{3}, \\ \frac{4x}{5} + \frac{2y}{7} = \frac{76}{35}. \end{cases}$$

$$26. \begin{cases} \frac{5}{m+n} + \frac{2}{m-n} = \frac{9}{4}, \\ \frac{9}{m-n} + \frac{10}{m+n} = 7. \end{cases}$$

$$27. \begin{cases} \frac{2}{x} + \frac{3}{y} = -\frac{1}{2}, \\ \frac{5}{x} + \frac{2}{3y} = -\frac{14}{3}. \end{cases}$$

$$28. \begin{cases} \frac{2}{3p} - \frac{5}{2n} = -\frac{43}{9}, \\ \frac{7}{2p} + \frac{10}{3n} = -\frac{11}{2}. \end{cases}$$

EXERCISES

Solve the following systems involving decimals, and check the sets of roots found:

$$1. \begin{cases} 2m + .3n = .82, \\ n - 4m = -2.6. \end{cases}$$

$$2. \begin{cases} 2x + 5y = 43, \\ 3x - .2y = 10.6. \end{cases}$$

$$3. \begin{cases} .13r - .24s = -.545, \\ .5r + .61s = .97. \end{cases}$$

$$4. \begin{cases} .3x - .5y = .4, \\ .2x + .4y = 1. \end{cases}$$

$$5. \begin{cases} .2x + .3y = .75, \\ .5x + .6y = 1.8. \end{cases}$$

$$6. \begin{cases} .8x + .2y = .7, \\ .4x - 2.0y = 10.15. \end{cases}$$

$$7. \begin{cases} .06x + .67y = 4.32, \\ .9x - 2y = -7.5. \end{cases}$$

$$8. \begin{cases} .8x + .33y = -2.4, \\ .1x - .66y = -.2. \end{cases}$$

$$9. \begin{cases} 4x + 1.5y = 9.6, \\ 5x - 3.5 + 4y = 19.125. \end{cases}$$

$$10. \begin{cases} 3x + 13y = 74, \\ 12x - 5.3y = -47.8. \end{cases}$$

$$11. \begin{cases} .2x + 5y = 15.6, \\ 3x - 2.5y = -6.25. \end{cases}$$

$$12. \begin{cases} \frac{1}{x} + \frac{1}{y} = 6, \\ \frac{1}{x} - \frac{1}{y} = -\frac{1}{.5}. \end{cases}$$

$$13. \begin{cases} \frac{5}{x} + \frac{6}{y} = 4, \\ \frac{7}{x} + \frac{1}{y} = 1.9. \end{cases}$$

$$14. \begin{cases} \frac{.2}{x} - \frac{.3}{y} = \frac{1.6}{6}, \\ \frac{5}{x} + \frac{.5}{y} = \frac{1}{3}. \end{cases}$$

$$15. \begin{cases} \frac{2}{x} + 10y = -39.6, \\ \frac{3}{x} - 5y = 20.6. \end{cases}$$

$$16. \begin{cases} 25x + .25y + \frac{117}{8} = 4(x+y), \\ \frac{x-y}{5} + 1.8 = \frac{y}{2}. \end{cases}$$

$$17. \begin{cases} 2x + \frac{.5}{y} = 5.5, \\ \frac{3.2}{y} - .4x = -4.4. \end{cases}$$

$$18. \begin{cases} 2x - 3y = -1, \\ 5.5x + 3.2y = 8.7. \end{cases}$$

$$19. \begin{cases} 15a - 2.5b = 4.75, \\ .3a - .4b = .08. \end{cases}$$

$$20. \begin{cases} \frac{.2}{x} - \frac{.5}{y} = 2, \\ \frac{.3}{x} + \frac{.1}{y} = 4.7. \end{cases}$$

In the following problems the student should state *two* equations in *two* unknowns. Instead of using x and y , the *first letter* of the word denoting an unknown should be used to represent that unknown wherever possible. Thus in Problem 7, p. 265, d should represent the number of dimes and q the number of quarters. The plan here suggested is desirable for many reasons, and should be followed in all problems containing two or more unknowns unless the words denoting two of the unknowns begin with the same letter.

In solving problems like 1-39 on pages 175-179 there are really two unknowns involved; but one of the equations to which each of those problems leads is so simple that what amounts to the method of substitution was employed by expressing one unknown in terms of the other.

EXAMPLE

The difference between two numbers is 17, and their sum is 33. Find the numbers.

Solution.

Let x = one number,

and

y = the other number.

Then

$$x - y = 17, \quad (1)$$

and

$$x + y = 33. \quad (2)$$

Solving for x and y , we get

$$x = 25,$$

$$y = 8.$$

PROBLEMS

1. The difference between two numbers is 25 and their sum is 41. Find the numbers.

2. The difference between two numbers is 3. Their sum is 11. Find the numbers.

3. What are the two numbers whose sum is 10 and whose difference is 6?

4. Find the two numbers whose sum is 5 and whose difference is 25.

5. The value of a certain fraction is $\frac{1}{2}$ when 1 is added to the numerator. When 4 is subtracted from the denominator the value becomes $\frac{3}{4}$. Find the fraction.

6. The value of a certain fraction is $\frac{1}{2}$. If 2 is added to the denominator and 1 is subtracted from the numerator the fraction becomes equal to $\frac{1}{3}$. What are the values of the numerator and denominator of the fraction?

7. The value of a collection of quarters and dimes containing 100 coins is \$10.90. How many coins of each kind are there?

8. There are two kinds of coins in a collection. When the collection contains 8 of one kind and 5 of the other it is worth \$1.65. If there are 10 of the first kind and 13 of the second kind, the lot amounts to \$3.75. What are the denominations of the two types of coins?

9. The difference of the numerator and denominator of a certain fraction is 1. Subtracting 5 from the numerator and adding 4 to the denominator makes the fraction equal to $\frac{1}{2}$. What is the original value of the fraction?

10. Two boys on a seesaw balance when one is 6 feet from the fulcrum and the other is 4 feet from the fulcrum. A third boy weighing 70 pounds joins the first boy, and the two balance the second boy in his original position when they are $2\frac{1}{2}$ feet from the fulcrum. What are the weights of the first and second boys?

11. The sum of two weights is 16 pounds. They balance each other when they are $\frac{1}{2}$ foot and $1\frac{1}{2}$ feet from the fulcrum respectively. Find the weights.

~~12.~~ A man has \$5000 invested, partly in a savings bank paying 4% interest and partly in bonds paying 6%. His income from the two sources is \$280 a year. How much has he invested in each place?

13. Part of \$15,000 is invested at 5% and the remainder at 4%. The income from the 4% investment is twice that from the 5%. How much is invested at each rate of interest?

14. A is now $\frac{2}{3}$ as old as B. Five years ago B was twice as old as A. What are their ages now?

15. In 10 years C will be $\frac{4}{3}$ as old as D. In 5 years D will be $\frac{5}{7}$ as old as C. What is the present age of each?

16. A man has \$2000 to invest, part at 7% and the remainder at 4%. He wishes an income of \$125 per year from this money. How much must he invest at each figure?

17. The sum of the digits of a two-digit number is 7. The number plus 27 is the original number with digits reversed. Find the original number.

Solution. Let t = the digit in tens' place,
and u = the digit in units' place.

$$\text{Then } t + u = 7. \quad (1)$$

But t standing in tens' place has its numerical value multiplied by 10. Therefore the number is represented by the binomial $10t + u$, and the number formed by the digits in reverse order is represented by the binomial $10u + t$.

$$\text{Hence } 10t + u + 27 = 10u + t. \quad (2)$$

$$\text{Simplifying (2), } u - t = 3. \quad (3)$$

$$\text{Solving (1) and (3), } t = 2 \text{ and } u = 5.$$

Hence the number is 25.

18. The sum of the digits of a two-digit number is 6. The digits are reversed if 36 is added to the number. What is the original number?

19. In a certain two-digit number the tens' figure is twice the units' figure. If the number with digits reversed is subtracted from the original number the remainder is 18. Find the original number. $(10x + 2x) - (2x + x) = 18 \Rightarrow 7x = 18 \Rightarrow x = \frac{18}{7}$

20. A certain number divided by $\frac{1}{2}$ the sum of the digits gives 17. If 36 is subtracted from the number, the remainder is the original number with digits reversed. Find the original number. $10x + y = 17(2x + y)$
 $10x + y = 34x + 17y \Rightarrow -24x = 16y \Rightarrow -3x = 2y \Rightarrow x = -\frac{2}{3}y$
 $10(-\frac{2}{3}y) + y = 17(-\frac{4}{3}y) \Rightarrow -\frac{20}{3}y + y = -\frac{68}{3}y \Rightarrow -\frac{17}{3}y = -\frac{68}{3}y \Rightarrow 51y = 68y \Rightarrow 17y = 0 \Rightarrow y = 0$
 $x = 0$

21. A certain two-digit number plus 8 is equal to ten times one less than the sum of the digits. The tens' digit is twice the units' digit. What is the original number?

22. The sum of a certain two-digit number and the sum of the digits of the number is 15. The number minus the sum of the digits is 9. Find the number. $10x + y + x + y = 15$
 $10x + y - x - y = 9 \Rightarrow 9x = 9 \Rightarrow x = 1$
 $10(1) + y + 1 + y = 15 \Rightarrow 11 + 2y = 15 \Rightarrow 2y = 4 \Rightarrow y = 2$
The number is 12.

The *reciprocal* of a number is a fraction of which the numerator is 1 and the denominator is the number itself.

Thus $\frac{1}{2}$ and $\frac{1}{a}$ are the reciprocals of 2 and a respectively.

23. What are the reciprocals of 3, 5, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{5}$, 1, $1\frac{1}{2}$, $8\frac{7}{10}$?

24. The sum of the reciprocals of two numbers is $\frac{7}{10}$, and the difference of the reciprocals is $\frac{3}{10}$. Find the numbers.

25. The sum of the reciprocals of two numbers is $\frac{7}{12}$. The greater reciprocal over the lesser reciprocal is equivalent to $1\frac{1}{3}$. Find the numbers.

26. A gives B \$50. B then has four times as much as A. Then B gives A \$100 and has left twice as much as A. How much money did each have in the first place?

27. A man has two barrels of gasoline. He takes 10 gallons from the first and puts it in the second. The second then contains three times as much as the first. Next he takes 30 gallons from the second and puts it in the first. The second barrel then contains $\frac{6}{10}$ as much as the first barrel. How much gasoline was there originally in each barrel?

28. A man walks in a railway train in the direction of the motion of the train for 2 minutes. He has then traveled with respect to the ground $\frac{1\frac{3}{8}}{30}$ of a mile. He returns through the train at the same speed to his starting point, and has traveled in returning $\frac{7}{30}$ of a mile with respect to the ground. What was his rate of walking in miles per hour, and how fast was the train moving?

29. A man rows a boat 5 miles downstream in 1 hour and returns in $1\frac{1}{2}$ hours. What is his speed? How fast is the current in the river flowing?

30. A man travels 100 miles, 50 miles by train and 50 miles by autobus. He makes the trip in 2 hours and 40 minutes. On the return trip he travels 30 miles by train and 70 miles by autobus, and takes 2 hours and 56 minutes. What was the speed of the train? of the autobus?

31. A man must make a trip of 200 miles. He can go part way by auto and the rest of the way by train. If he takes a bus going 20 miles per hour to the station and takes the train the remainder of the distance he can make the trip in $5\frac{5}{8}$ hours. If he takes a taxi traveling 30 miles per hour to the same station and the train for the remainder of the way he can make his journey in $5\frac{5}{4}$ hours. What is the distance to the station? How far does he go by train? What is the speed of the train?

32. In Problem 31 if the train fare is 4¢ per mile, the bus fare 5¢ per mile, the taxi fare 40¢ per mile, and the man's time is worth \$5 per hour, what is the most economical way for him to travel? How much would his time have to be worth to make it economical for him to take the taxi to the train?

33. Two men, A and B, can do a piece of work together in 4 days. They work together on the job for $1\frac{1}{2}$ days and then B finishes it alone in $7\frac{1}{2}$ days more. How many days would each man require to do the work alone?

34. A boat going 18 miles per hour travels downstream from A to B in 2 hours. It then returns to a point, C, 7 miles below A, in 3 hours. What is the distance from A to B? from B to C? What is the speed of the current?

106. Literal equations in two unknowns. Linear systems in which the unknowns have literal coefficients are usually solved by the method of addition and subtraction.

EXERCISES

In the following exercises consider a, b, c, d , and l as known numbers; solve for the other letters involved and check:

1. $x + y = 4a,$
 $x - 2y = a.$

2. $x + y = 5a,$
 $x - y = a.$

3. $m + n = c,$
 $m - n = d.$

4. $3x + 4y = b,$
 $5x + 3y = c.$

$$2x + 3cy = 3c,$$

5. $y - x = -\frac{3c}{2}.$

$$\frac{2x}{3} + \frac{9y}{d} = 18,$$

6. $3dx - \frac{2y}{d} = -4.$

$$\frac{9x}{4} + \frac{3y}{5} = \frac{123a}{20},$$

7. $\frac{2x}{a} + \frac{5y}{2a} = \frac{7}{2}.$

- $$\begin{array}{ll} \frac{2x}{3} - \frac{5y}{6} = a, & x + by = a, \\ 8. \frac{x}{2} + \frac{y}{4} = -a. & 16. 2x - ay = \frac{a}{b}(4b + a). \\ 9. 0.25x + 0.6y = .45a, & \frac{1}{x} + \frac{1}{y} = a, \\ 0.5y - 0.4x = 0.01a. & 17. \frac{2}{x} - \frac{1}{y} = b. \\ 10. \frac{2v}{5} + \frac{5s}{3} = -4.2a, & \frac{a}{y} + \frac{b}{x} = d, \\ \frac{v}{2} - \frac{s}{3} = 2a. & 18. \frac{a}{y} - \frac{b}{x} = c. \\ 11. 4x + 3y = 22a, & 19. cm + dn = d + c, \\ 5x + 7y = 47a. & dn - cm = d - c. \\ xa + y = a, & 20. ax - cy = b, \\ 12. \frac{x}{2} + \frac{y}{a} = 0. & 4ax - 3cy = a + 3b. \\ 13. ax + 2y = -\frac{4}{3}x - 3a^2, & 21. bz + cw = d, \\ 3ay - 2x = 6(a^2 + a). & lw + z = a. \\ 14. 3m + 2n = (a + b), & .5m + .2n = .2, \\ 12m + 9a = 4b + 18n. & 22. 1.2n - .3m = -2.76. \\ 15. x - 3y = 2b, & .2x - .3y = -.25l, \\ 3x + 2y = 17b. & 23. .5x + 1.3y = 1.65l. \end{array}$$

GENERAL ORAL EXERCISES

1. If 1 book costs $2a$ dollars, how much will d books cost?

2. If 3 books cost b dollars, how much will n books cost?

3. What is the perimeter of a rectangle of length l inches and breadth b inches? What is its area?

4. If n books cost d dollars, what will one book cost?
 b books?

5. The area of a rectangle is ab square inches. If the length is a feet, what is the width? the perimeter?

6. The length of a rectangle is a feet. The width is $b - a$ feet. What is the area? the perimeter?

7. A triangle has an altitude a feet and a base $b - l$ feet. What is the area?

8. A triangle has an altitude h feet and a base equal to $2b$ inches. What is its area?

9. A triangle has an altitude of l feet. If the altitude is $\frac{1}{2}$ the base in feet, what is the base? the area?

10. Two trains start from the same station and travel in opposite directions a and b miles per hour respectively. How far will each be from the station in t hours? How far apart will they be at that time?

11. If these trains travel in the same direction, how far apart will they be from each other in t hours? How far will each be from the station from which they started?

12. The sum of two numbers is 10, and the lesser is s . What is the greater? What is the difference? the product?

13. If x is A's age now, what does $(x - 5)$ indicate? What does the equation $x + 2 = 2(x - 5)$ mean?

14. A man receives d dollars per working day for 4 weeks. In this period he spends e dollars. How much money has he at the end of 4 weeks of 6 working days each?

15. A man can walk m miles in 1 hour. How far can he go in h hours? How long will it take him to walk d miles?

16. It costs a man d dollars a day to live. How long can he live on p dollars? How much money will it take to support n men one day? t days?

17. If 9 apples can be bought for c cents, how many can be had for d cents? How much will a apples cost?

18. A triangle has an area a square inches and base b feet. What is its altitude?

19. A farmer has fodder for 12 cattle n days. How long can m cattle live on the same amount of food?

20. One man does a piece of work in n days. How much can he do in d days? How long will it take m men to do the job if each man works at the same rate as the first man?

GENERAL PROBLEMS

1. The base of a triangle is b inches and the altitude is 6 inches. If the base is increased 3 inches, how much must the altitude be reduced to keep the area the same?

HINT. Let x = the decrease of the altitude in inches.

Then $6b/2 = (b + 3)(6 - x)/2$, etc.

2. The altitude of a triangle is a inches and the base is b inches. If the base is increased by n inches, how much must the altitude be reduced to keep the original area?

3. A rectangle has a length of l inches and a breadth of b inches. If the length is reduced d inches, how much must the breadth be increased to keep the area the same?

4. The length and width of a rectangle are 6 and 4 inches respectively. The length is decreased by the same amount as the width is increased, while the area remains the same. How much is the length decreased?

5. The sum of two numbers is s and their difference is d . What are the numbers?

6. The sum of two numbers is q and their difference is one half of q . What are the numbers?

7. A certain number is 3 larger than another number. The quotient of the smaller divided by the larger is q . Find the numbers.

8. The value of a fraction is a . If 1 is added to the numerator the value becomes b . Find the numerator and the denominator.

9. If b is added to the numerator of a certain fraction, the resulting fraction is equal to 3. If a is added to the denominator the fraction is equal to 1. Find the numerator and the denominator.

10. If 2 is added to the numerator of a certain fraction, the resulting expression is equal to x . When 3 is subtracted from the denominator the fraction is equal to z . What are the values of the numerator and the denominator?

11. A collection of eggs contains a certain number costing \$1 per dozen and another number costing 70¢ per dozen. Together the eggs cost b dollars. If the number of eggs of each grade had been interchanged, the cost of the whole would have been a dollars. How many dozen eggs were there of each grade?

12. A boy weighing p pounds balances a boy weighing w pounds on a seesaw. If the distance between the boys is l , what is the distance of each boy from the fulcrum?

13. A and B have together d dollars. A has m dollars more than B. How much has each?

14. If A gave g dollars to B, he would then have half as much as B. If B gave k dollars to A, he would have 1 dollar more than A. How much money has each?

15. Together A and B have \$50. A gives g dollars to B and then B gives f dollars to A, whereupon they have equal amounts. How much money did each have?

16. A man has d dollars invested, part at 4% and part at 5% interest. His income is m dollars per year. How much money has he invested at each rate of interest?

17. Two men, A and B, start from points m miles apart and walk toward each other. A walks at r miles per hour, and B walks at s miles per hour. How long will they have to walk before they meet? How far will they be from their respective starting points when they meet each other?

18. A works half as fast as B. The two working together can do a piece of work in d days. How long would it take each man alone to do it?

19. A does a job in c days. A and B together do the job in d days. How many days would it take B to do the job alone?

20. A works four times as fast as B. The two together do a piece of work in p days. How long will it take each man alone to do the same job?

21. A and B together do a job in d days. If A works t times as fast as B, how long would it take each to do the work?

REVIEW EXERCISES

1. Solve $4(10 - 2x) - 3(x - 15) = 0$.

2. Solve $\frac{5x + 3}{4} = 7$.

3. Solve the system $\begin{cases} 2x + 5y = 24, \\ 3x - y = 2. \end{cases}$

4. Solve $\frac{3x - 5}{x - 3} + \frac{2x - 5}{x - 4} = \frac{35(x - 2)}{7x - 24}$.

5. Solve the system $\begin{cases} \frac{3x}{2} - \frac{5y}{3} + 2 = \frac{1}{6}, \\ \frac{2x}{3} + \frac{5y}{3} = \frac{17}{3}. \end{cases}$

Factor:

6. $a^2 + 2ab - 15b^2$. 7. $(a-b)^2 - x^2$. 8. $x^3 + 4x^2 + 4x$.

Reduce to a common denominator and simplify:

9. $\frac{1}{a-b} - \frac{1}{a+b}$.

10. $\frac{8}{2x-3} + \frac{5}{3-2x} - \frac{3x-4}{2x^2-x-3}$.

Divide:

11. $a^3 - a^2b + 2b^3$ by $a + b$.

12. $6x^3 + x^2 - 29x + 21$ by $2x - 3$.

Solve for x and then find its numerical value for the given values of the other letters:

13. $ax + x = m$, when $a = 2$ and $m = -3$.

14. $a(x-1) - b = x - a$, when $a = 2$ and $b = 0$.

15. $ax + bx = m + x$, when $a = -4$, $b = \frac{1}{2}$, and $m = 4$.

16. $m(a+b-x) = n(a+b-x)$, when $a = -1$, $b = 2$, $m = -3$, and $n = 5$.

17. The sum of two numbers is 51. Their difference is 13. Find the two numbers.

18. Two boys weigh together 130 pounds. They balance on a seesaw when they are 3 feet and $3\frac{1}{2}$ feet from the fulcrum respectively. What are their weights?

19. A collection of nickels and quarters is worth \$2.40 and contains 20 coins. What is the number of each kind of coin in the collection?

20. A and B had between them \$1000. A spent $\frac{1}{2}$ of his share while B spent $\frac{1}{3}$ of his. If A spent \$25 more than B, what did each have in the first place?

CHAPTER XIX

GRAPHS OF EQUATIONS

107. Introduction. In Chapter II several devices were given for the graphic presentation of statistics. One of these, the line graph, is similar in its general features to the method employed for representing geometrically the relation between sets of numbers which are connected by an algebraic equation.

The question What two numbers added give seven? may be expressed by the equation $x + y = 7$. Here x and y are any two numbers whose sum is 7. It can be seen by inspection that if $x = 4$, $y = 3$; if $x = 1$, $y = 6$; if $x = 0$, $y = 7$; etc. We may also assign to x any value, say -2 ; then the equation becomes $-2 + y = 7$, whence $y = 9$. In this manner we may obtain an unlimited number of sets of related values for x and y , some of which are given in the following table:

$$x + y = 7$$

	<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>	<i>E</i>	<i>F</i>	<i>G</i>	<i>H</i>	<i>I</i>
If $x =$	5	1	0	6	7	- 1	- 2	8	9
then $y =$	2	6	7	1	0	8	9	- 1	- 2

108. Definitions and assumptions. In constructing the graph of an equation in two variables a number of assumptions must be made. These assumptions and some necessary definitions are now stated.

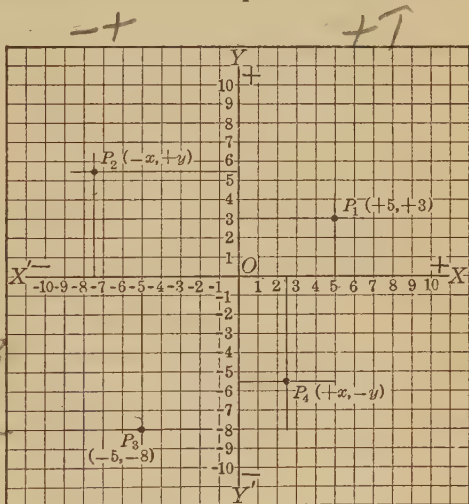
It is agreed :

I. To have two lines at right angles to each other, — as $X'OX$, called the x -axis, and $Y'OY$, called the y -axis, as in the figure below.

II. To have a line of definite length as a unit of distance. Then the number 2 will correspond to a distance twice the unit in length, the number $4\frac{1}{2}$ to a distance of $4\frac{1}{2}$ times the unit, etc.

III. That the distance (measured parallel to the x -axis) from the y -axis to any point in the surface of the paper be the x -distance (or abscissa) of that point, and the distance (measured parallel to the y -axis) from the x -axis to the point be the y -distance (or ordinate) of the point.

IV. That the x -distance of a point to the *right* of the y -axis be represented by a *positive* number, and the x -distance of a point to the *left* by a *negative* number ; also that the y -distance of a point *above* the x -axis be represented by a *positive* number, and the y -distance of a point *below* the x -axis by a *negative* number. Briefly, *distances measured from the axes to the right or upward are positive, to the left or downward are negative.*



V. That every point in the surface of the paper corresponds to a *pair of numbers*, one or both of which may be positive, negative, integral, or fractional.

VI. That of a given pair of numbers the first be the measure of the x -distance and the second the measure of the y -distance. Thus the point $(5, 3)$, or P_1 , in the figure, is the point whose x -distance is 5 and whose y -distance is 3. Again, the point $(-5, -8)$, or P_3 , is the point whose x -distance is -5 and whose y -distance is -8 .

The point of intersection of the axes is called the *origin*.

The values of the x -distance and the y -distance of a point are often called the *coördinates* of the point.

Though not an absolute necessity, cross-ruled paper is a great convenience in all graphical work. Excellent results, however, can be obtained with ordinary paper and a rule marked in inches and fractions of an inch for measuring distances. Hence the graphical work which follows should not be omitted even though it is found inconvenient to obtain cross-ruled paper for class use.

EXAMPLE

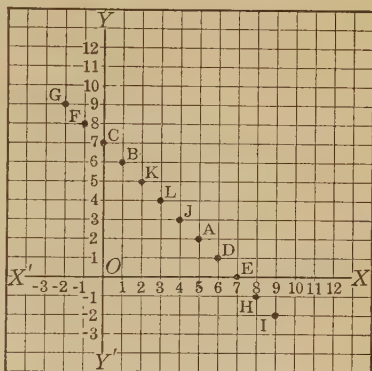
Using $\frac{1}{4}$ inch for the unit of measure, locate points corresponding to the coördinates which follow: $A, (5, 2)$; $B, (1, 6)$; $C, (0, 7)$; $D, (6, 1)$; $E, (7, 0)$; $F, (-1, 8)$; $G, (-2, 9)$; $H, (8, -1)$; $I, (9, -2)$; $J, (4, 3)$; $K, (2, 5)$; and $L, (3, 4)$.

Solution. According to VI, above, the x -distance of the point $(5, 2)$ is 5 and its y -distance is 2. Hence to locate point A at $(5, 2)$ we measure, according to IV, five units to the right of the origin on the x -axis and from that point two units upward.

To locate point F at $(-1, 8)$ we measure on the x -axis one unit to the left of the origin and from that point upward eight units. Point F thus located corresponds to $(-1, 8)$.

Points for the other pairs of numbers given in the example should be located by the pupil. The correct positions for these points can be seen in the figure.

Locating points as in the example above is called *plotting points*.



EXERCISES

Plot the following points with reference to the two axes. It is suggested that the unit of measure be $\frac{1}{2}$ inch.

1. $(5, 1)$; $(3, 4)$; $(3.25, 0)$; $(1, 2)$; and $(0, 0)$.
2. $(-1.5, 2)$; $(1, -3)$; and $(2.5, 4)$.
3. $(-2, -4)$; $(1.5, -0.5)$; $(-1, -1)$; and $(0, -3)$.
4. $(5, -2)$; $(-1, -3.5)$; $(2.25, -3)$; $(1.75, -6)$; and $(0.75, 3.25)$.

5. If the x -distance of a point is zero, where is the point located? Where is it located if both of its coördinates are zero?

109. The graph of an equation. On page 276 we computed several sets of values of x and y for the equation $x + y = 7$. Later these points were plotted in locating A, B, C, D, E, F, G, H , and I of the preceding example. It

is evident from an inspection of their position that a straight line can be made to pass through all the points there located. The *line* drawn through these points is said to be the *graph of the equation* $x + y = 7$.

EXERCISES

1. Find and tabulate six pairs of values of x and y which satisfy the equation $x + 2y = 6$. Draw two axes and, using $\frac{1}{2}$ inch as the unit distance, plot each of the points. Are the six points in a straight line? Does $x = 3$, $y = 3$ satisfy this equation? Plot the point $(3, 3)$. Is it on the graph of the equation? If the x - and y -distances of a point satisfy the equation $x + 2y = 6$, where is the point located? If the x - and y -distances of a point do not satisfy the equation $2x + y = 8$, what can be said of its location?

Find and tabulate six pairs of values for x and y which satisfy each of the following equations. Use numbers not greater than 10. Use at least one negative value for x and one negative value for y . Then plot the six corresponding points. Can a straight line be drawn through the six points obtained in each exercise?

2. $x - 5y = 6$. 4. $3x - 2y = 12$. 6. $3x = 2y$.

3. $x - 2y = 4$. 5. $2x - y = 0$. 7. $y = 2x - 1$.

110. **Equation of a straight line.** The preceding work should convince the student that the graph of an equation of the first degree in x and y is a *straight line*. This fact can be proved, but the proof is too difficult to be given at present. Therefore it will be assumed that the graph of every linear equation in two variables is a straight line. And since the position of a straight line is known when

any two of its points are given, it will be sufficient in graphing a linear equation in two variables to plot *any two points* whose x - and y -distances satisfy the equation, and then to draw a straight line through these two points. The two points most convenient to plot are usually the two in which the line cuts the axes. Occasionally these points come very close together, and consequently they will not determine accurately the position of the line. In such cases one should decide on two values of x rather far apart (such as 0 and 5, or 0 and -5) and compute the corresponding values of y . Two such points will fix the position of the line with sufficient accuracy.

If a line goes through the origin (as in Exercise 5 preceding), $x = 0$, $y = 0$ will do for one point, but a point not on both axes must be taken for the second one.

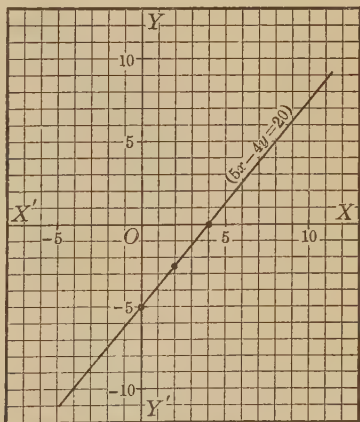
The essentials of the method of graphing a given linear equation in x and y are illustrated in the following

EXAMPLE

Graph the equation

$$5x - 4y = 20.$$

Solution. In this equation, if $x = 0$, $y = -5$; and if $y = 0$, $x = 4$. Here the point $(0, -5)$ is on the y -axis in the adjacent figure, 5 units below the origin; and the point $(4, 0)$ is on the x -axis, 4 units to the right of the origin. The straight line through these two points is the graph of $5x - 4y = 20$.



The necessary work may be tabulated as follows :

$$5x - 4y = 20.$$

If $x =$	0	4	2
then $y =$	- 5	0	$-2\frac{1}{2}$

Check. If an error has been made in obtaining the value of x or y from the equation, or in plotting the values found, it can be quickly detected by plotting a third point, as $(2, -2\frac{1}{2})$, the values of whose x - and y -distances satisfy the equation. If this third point lies on the line determined by the first two points, the line is probably correct; if it does not, an error has been made.

EXERCISES

Graph the following linear equations :

- | | | |
|-------------------|-------------------|------------------|
| 1. $x + y = 7.$ | 4. $x - y = 0.$ | 7. $x - 2y = 0.$ |
| 2. $x + y = 0.$ | 5. $3y + 2x = 4.$ | 8. $x + y = -5.$ |
| 3. $5x - 2y = 9.$ | 6. $4x + y = 3.$ | 9. $x = 5.$ |

HINT. The equation $x = 5$ is equivalent to the equation $x + 0y = 5$. The latter is satisfied by $x = 5$, and *any* value of y . Thus the pairs of values $(5, 1)$, $(5, 5)$, $(5, -2)$, etc. satisfy the equation $x + 0y = 5$. Plotting these points, it is evident that the required graph is a line parallel to the y -axis and 5 units to the right of it.

- | | | |
|---------------|---------------|--------------|
| 10. $y = -3.$ | 12. $x = 0.$ | 14. $y = 2.$ |
| 11. $x = 9.$ | 13. $x = -4.$ | 15. $y = 5.$ |

16. Is the point $(4, 3)$ on the line whose equation is $2x - 3y = 12$? Is $(0, 6)$? Is $(6, 0)$?

17. If a point is on a line, do the values of its x -distance and its y -distance satisfy the equation of the line?

18. If the values of the x -distance and the y -distance of a point satisfy the equation of a line, is the point located on the graph of the equation?

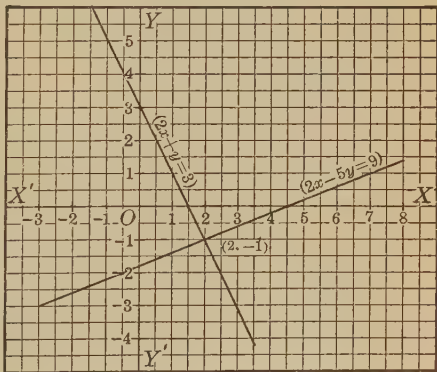
19. Determine without reference to the graph itself whether the point (6, 5) is on any of the graphs of the equations in Exercises 1–15 above. If so, on which does it lie?

It should now be clear that

The equation of a line is satisfied by the values of the x -distance and the y -distance of any point on that line.

Any point the values of whose x -distance and y -distance satisfy the equation is on the graph of the equation.

111. Graphical solution of linear equations in two variables. If we construct the graphs of the two equations $2x - 5y = 9$ and $2x + y = 3$ as indicated in the figure below, it is seen that for the point of intersection of the graphs x is 2 and y is -1 . Since the point (2, -1) is on both graphs, these values should satisfy both equations. Substituting 2 for x and -1 for y in each equation, we get the identities $4 - (-5) = 9$



and $4 + (-1) = 3$. Thus the *graphical solution* of two linear equations consists in plotting the two equations and finding from the graph the value of x and the value of y at the point of intersection. Since two straight lines can intersect in but one point, there can be but one pair of values of x and y which satisfies a pair of linear equations in two variables.

The necessary work is tabulated as follows:

$$2x - 5y = 9$$

$$2x + y = 3$$

If $x =$	0	$4\frac{1}{2}$	7
then $y =$	$-1\frac{4}{5}$	0	1

If $x =$	0	1	$1\frac{1}{2}$
then $y =$	3	1	0

EXERCISES

Solve graphically the following systems and verify by substituting in each set of equations the values of x and y for the point of intersection as obtained from the graph:

1. $4x - 3y = -5,$ $7x + y = 10.$
2. $x + y = 2,$ $3x - 2y = 16.$
3. $2x + 3y = 13,$ $x - y = 4.$
4. $4x + y = 17,$ $x - y = -2.$
5. $14y + 12x = 13,$ $2x + 6y = -7.$
6. $4x + 7y = 5,$ $18y - 5x = -33.$
7. $x + 3y = 5,$ $2x - 5y = -1.$
8. $x + 8y = 16,$ $7y - 5x = 14.$
9. $2x + 2y = 3,$ $4x - 6y = 1.$
10. $3x - 2y = 0,$ $6x - 4y = 7.$
11. $x = 1,$ $9x + 8y = 17.$
12. $x = 4y,$ $2x - 3y = 5.$

BIOGRAPHICAL NOTE

RENÉ DESCARTES. One of the two or three most important advances ever made in mathematics was the discovery that algebraic equations could be represented geometrically. This great discovery was made by René Descartes (1596-1650), the French philosopher. Though never rugged in health, he took part in several campaigns when a young man, and it is said that during a weary winter spent in camp in Austria he first conceived the ideas that resulted in this important work. Though his writings read very differently from a modern book on the same subject, yet he developed all the essentials of graphical representation. He saw that a letter, that is, a coördinate, might represent either a positive or a negative number, and so forced upon mathematicians the conviction that negative integers are indeed numbers and that they are useful in algebraic operations. After his

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1970



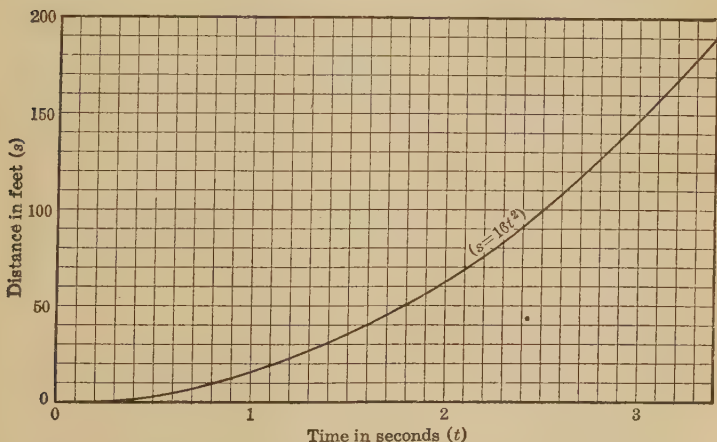
René Descartes

time they were not usually ruled out as absurd or impossible, as was commonly the case before. He also introduced the modern exponential notation. To Descartes is due the use of the last letters of the alphabet for the unknown and the first letters for the known numbers. Though the sign $=$ was used long before his time, he did not accept it. He used the asterisk to indicate that a certain power of the variable was lacking. Thus he would have written the equation $x^3 - 8x + 16 = 40$ in the form $x^{3*} - 8x + 16 \infty 40$.

112. Graphs of formulas. In many cases it is possible to use a graph as a substitute for numerical computation.

EXAMPLE

If a body falls from rest (neglecting the resistance of the air), the distance in feet, s , which it travels in t seconds is given by the formula $s = 16t^2$. Graph the formula.



From this graph estimate how far the body falls in $1\frac{1}{2}$ seconds; in $3\frac{1}{2}$ seconds. How long does it take for the body to fall 8 feet? 36 feet? 150 feet?

EXERCISES

1. One dollar at simple interest for t years at 6% will amount to a sum represented by the expression $A = 1 + .06t$. Draw the graph for this formula. From the graph tell how long it would take a dollar to double itself at this rate of interest. If a boy at the age of 10 had deposited a dollar in the bank at 6% interest, to how much would his account amount by the time he was 35 years of age?

2. A dollar at 6% interest compounded annually for a period of n years will increase in accordance with the formula $A = (1 + .06)^n$. Plot this formula. How long would it take a dollar to double under these conditions? If a boy 10 years old deposited one dollar in the bank at 6% interest compounded annually, how much would it amount to if he left it there from the time he was 10 years old until he was 35?

3. The circumference of a circle is expressed by the formula $C = 2\pi r$, where $\pi = 3.14$. Graph this equation. From the curve estimate the circumference of a circle which is 10 feet in diameter; 3 feet in diameter. A circle has a circumference of 12 feet; what is its radius?

4. The area of a circle is given by the expression $A = \pi r^2$. Graph this equation. If a circle has an area of 25 square feet, what is its radius? What is the area of a circular table-top $3\frac{1}{2}$ feet in diameter?

5. A falling body moves with a velocity represented by the formula $V = 32t$, where V is the velocity of the body in feet per second and t is the time from rest expressed in seconds. Graph this equation. From the graph find the time it will take before a body moves with a speed of 50 feet per second; 82 feet per second. How fast is the body moving after it has been falling for 5 seconds? 13 seconds? $\frac{1}{2}$ second?

6. The velocity of a jet of water flowing from a hole in a sheet-metal tank is expressed by the formula $V = 8\sqrt{H}$, where V is the velocity of the jet in feet per second and H is the height in feet (often called the "head") of the water level in the tank above the hole. Graph this formula for various heads up to 100 feet. By inspection of the curve tell what is the velocity when the head is 20 feet; 45 feet; 33 feet. What must the head be to give a velocity of 10 feet per second? $22\frac{1}{2}$ feet per second?

HINT. Substitute values 0, 1, 4, 9, 16, 25, 36, etc. for H in determining the corresponding values of V .

7. The horse power of an engine is expressed by the approximate formula $HP = \frac{nb^2}{2.5}$, where HP is the horse power, b the bore (or diameter) of one of the cylinders in inches, and n the number of cylinders. Graph this expression in the case of a six-cylinder engine. How many horse power would you get from a six-cylinder engine with a 1-inch cylinder? a 6-inch cylinder? What diameter of cylinder would you need to develop 100 HP ? 25 HP ?

8. The horse power required to drive an automobile over a smooth, level road against the resistance of the air is given by the following formula:

$$HP = \frac{0.0185V^3 + 147V}{550}.$$

In this equation HP is the required horse power and V is the speed of the car in miles per hour. Graph this formula and determine the following values: What would be the required power to drive the car 25 miles per hour? 100 miles per hour? What will be the maximum speed obtainable with a 50- HP motor? a 10- HP motor?

CHAPTER XX

SQUARE ROOT

113. Square root of algebraic expressions. Evidently

$$t^2 + 2tu + u^2 = [\pm(t + u)]^2.$$

A study of this form will enable us to extract the square root of any polynomial. Obviously, the square root of t^2 (the first term of the trinomial) is t , the first term of the root. If t^2 is subtracted from the trinomial, the remainder is $2tu + u^2$. The next term of the root (u) can be found by dividing the first term of the remainder ($2tu$) by $2t$ (twice that term of the root already found).

The work may be arranged thus:

$$\begin{array}{r} t + u \\ \hline t^2 + 2tu + u^2 \\ \hline t^2 \\ \hline \text{Trial divisor, } 2t \quad 2tu + u^2 \\ \text{Complete divisor, } 2t + u \quad 2tu + u^2 = (2t + u)u \end{array}$$

Therefore the required roots are $\pm(t + u)$.

The foregoing process is easily extended to the polynomial $9x^4 - 24x^3 + 28x^2 - 16x + 4$, as follows:

$$\begin{array}{r} 3x^2 - 4x + 2 \\ \hline 9x^4 - 24x^3 + 28x^2 - 16x + 4 \\ \hline (3x^2)^2 = 9x^4 \quad \quad \quad -24x^3 + 28x^2 \\ \hline \text{First trial divisor, } 2 \cdot 3x^2 = 6x^2 \quad \quad \quad -24x^3 + 16x^2 = (6x^2 - 4x)(-4x) \\ \text{First complete divisor, } 6x^2 - 4x \quad \quad \quad \hline \text{Second trial divisor, } \quad \quad \quad 12x^2 - 16x + 4 \\ \quad \quad \quad 2(3x^2 - 4x) = 6x^2 - 8x \quad \quad \quad \hline \text{Second complete divisor, } 6x^2 - 8x + 2 \quad \quad \quad 12x^2 - 16x + 4 = (6x^2 - 8x + 2)2 \end{array}$$

Therefore the required roots are $\pm (3x^2 - 4x + 2)$.

The term $3x^2$ was obtained by taking the square root of $9x^4$; the second term, $-4x$, by dividing $-24x^3$ by the first trial divisor, $6x^2$; and the third term, 2 , by dividing $12x^2$ by $6x^2$, the first term of the second trial divisor.

The method just illustrated of extracting the square root of a polynomial may be stated in the

RULE. *Arrange the terms of the polynomial according to descending or ascending powers of some letter in it.*

Extract the square root of the first term. Write the result (with plus sign only) as the first term of the root and subtract its square from the given polynomial.

Double the root already found for the first trial divisor, divide the first term of the remainder by it, and write the quotient as the second term of the root.

Annex the quotient just found to the trial divisor, making the complete divisor; multiply the complete divisor by the second term of the root and subtract the product from the last remainder.

If terms of the polynomial still remain, double the root already found for a trial divisor, divide the first term of the trial divisor into the first term of the remainder, write the quotient as the next term of the root, form the complete divisor, and proceed as before until the process ends, or until the required number of terms of the root have been found.

Inclose the root thus found in a parenthesis preceded by the sign $\pm$.

NOTE. The process of extracting the square root of numbers was familiar to mathematicians long before they knew how to find the square root of polynomials. This is consistent with the fact that the development of the methods of performing operations on literal number symbols generally followed and grew out of the similar operations on numerals. The application of the rules for extracting the square root of numbers to that of polynomials is generally ascribed to Recorde (1510-1558), who was the author of the earliest English work on algebra that we know. This book, which bears the title "The Whetstone of Wit," gives an accurate idea of the algebraic knowledge of the time and had a very wide influence.

EXERCISES

Obtain the positive square root of :

1. $a^2 + 4a + 4.$

4. $4x^2 - 20xy + 25y^2.$

2. $16 + 8x + x^2.$

5. $25 + 20a + 4a^2.$

3. $9m^2 + 12mn + 4n^2.$

6. $36n^6 + 36n^4 + 9n^2.$

7. $x^8 + 2x^4y^2 + y^4.$

8. $4s^4 + 4s^2t + 4s^2r + 2tr + t^2 + r^2.$

9. $9c^2 - 12cb^3 + 4b^6.$

10. $4a^2 + 4ab - 16ac + b^2 - 8bc + 16c^2.$

11. $25a^4 + 20a^3b + 94a^2b^2 + 36ab^3 + 81b^4.$

12. $36n^4 + 60n^5 + 36n^2 + 25n^6 + 30n^3 + 9.$

13. $36x^6 - 24nx^3 + 12n^2x^3 + 4n^2 - 4n^3 + n^4.$

14. $4a^2 + 20ab + 8ab^2 + 25b^2 + 20b^3 + 4b^4.$

15. $c^2 + d^2 + x^4 - 2dx^2 + 2cd - 2cx^2.$

16. $100r^2 + 20rc^2 - 20rs + c^4 - 2c^2s + s^2.$

17. $\frac{9x^2}{4} + xy + \frac{y^2}{9}.$

18. $16a^4 + \frac{4a^2b^2}{9} + \frac{16a^3b}{3}.$

19. $36n^2 - 9mn + 12m^2n + \frac{9m^2}{16} - \frac{3m^3}{2} + m^4.$

20. $\frac{25x^2}{4} + \frac{4y^2}{49} - \frac{10xy}{7}.$

21. $\frac{36m^6}{25} + \frac{16m^3n^2}{5} + \frac{16n^4}{9}.$

22. $\frac{r^2}{4} + \frac{rs}{4} - \frac{rt}{3} + \frac{s^2}{16} - \frac{st}{6} + \frac{t^2}{9}.$

Find the first three terms in the approximate square root of:

23. $4 + 8M + 12M^2$.

26. $25x^2 + 10x + 4$.

24. $25x^2 + 40x + 36$.

27. $25t^2 + 18t + 16$.

25. $36 - 14a + 22a^2$.

28. $9x^2 + 16x + 4$.

29. $100x^4 - 100x^3 + 68x^2 - 20x + 4$.

30. $4x^4 + 17x^2 - 12x + 4 - 12x^3$.

31. $m^4 - 6m^3 + 10m^2 + \frac{1}{4} - 3m$.

32. $9r^4 - 12r^3 + 5r^2 + 10 - 2r$.

114. Square root of arithmetic numbers. Since $1 = 1^2$, and $81 = 9^2$, a one-digit or a two-digit square has only *one* digit in its square root.

And as $100 = 10^2$, and $9801 = (99)^2$, a three-digit or a four-digit square has *two* digits in its square root.

Also, $10,000 = 100^2$, and $998,001 = (999)^2$; hence a five-digit or a six-digit square has *three* digits in its square root.

These examples illustrate the relation between the number of digits in a number and the number of digits in its square root. They also suggest a method of obtaining the first digit in the square root of any number.

For example, take the four numbers 35 24 18, 3 52 41, .25 38, and .05 28 6. Beginning at the decimal point in each, point off groups of two digits each, as indicated. Any incomplete group of two digits on the right, as in .05 28 6, should be completed by annexing one zero; thus, .05 28 60. Now the first digit in the square root is the greatest integer whose square is less than or equal to the left-hand group. This is true whether the latter contains

two digits or one. Hence the first digit in the square root of 35 24 18 is 5, in the square root of 3 52 41 is 1, in the square root of .25 38 is 5, and in the square root of .05 28 60 is 2.

Moreover, the number of digits in the square root of a perfect square is equal to the number of periods, provided a *single digit* remaining on the left is called a period.

115. Extraction of the square root of numbers. The square root of arithmetic numbers may be found by a procedure based on the method of extracting the square root of polynomials.

Just how t and u are involved in the square of $(t + u)$, or $t^2 + 2tu + u^2$, is obvious on inspection, because the parts t^2 , $2tu$, and u^2 cannot be united into one term. In the square of an arithmetic number, however, the parts are united. Thus $(47)^2 = (40 + 7)^2 = 1600 + 560 + 49 = 2209$. Now it is clear how 40 and 7 are involved in $1600 + 560 + 49$, but it is not plain from 2209 alone. Pointing off, however, enables us to discover at once the first digit, 4, which is equivalent to 4 tens, or 40. With the exception of pointing off, the method of extracting the square root of an arithmetic number does not differ greatly from the method of extracting the square root of an algebraic expression. In fact, the formula which states that the square root of $t^2 + 2tu + u^2 = \pm (t + u)$ can be used to explain the two processes.

If t denotes the tens and u the units, $t^2 + 2tu + u^2$ is closely related to $1600 + 560 + 49$, t^2 being 1600, or $(40)^2$; u^2 being 49, or 7^2 ; and $2tu$ being $2 \cdot 40 \cdot 7$. Therefore the process of extracting the square root of 2209 may be based on these relations and the work arranged as shown on the following page.

$$\begin{array}{r}
 \begin{array}{l}
 t^2 = (40)^2 \\
 2t = 2 \cdot 40 = 80 \\
 2t + u = 80 + 7
 \end{array}
 \begin{array}{r}
 \underline{4 \ 7, \text{ or } 40 + 7} \\
 \widehat{22 \ 09} \\
 16 \ 00 \\
 \hline
 6 \ 09 \\
 \hline
 6 \ 09 = (80 + 7)7 = (2t + u)u = 2tu + u^2
 \end{array}
 \end{array}$$

Therefore ± 47 are the two square roots of 2209.

If the number has three digits in its square root, the work and explanations may be arranged thus:

$$\begin{array}{r}
 \begin{array}{l}
 t^2 = (300)^2 \\
 \text{First trial divisor,} \\
 2t = 2 \cdot 300 = 600 \\
 \text{First complete divisor,} \\
 2t + u = 600 + 10 = 610 \\
 \text{Second trial divisor,} \\
 2t = 2 \cdot 310 = 620 \\
 \text{Second complete divisor,} \\
 2t + u = 620 + 5 = 625
 \end{array}
 \begin{array}{r}
 \underline{3 \ 1 \ 5, \text{ or } 300 + 10 + 5} \\
 \widehat{9 \ 92 \ 25} \\
 9 \ 00 \ 00 = 30 \text{ tens squared} \\
 \hline
 92 \ 25 \\
 \hline
 61 \ 00 = (2 \cdot 30 \text{ tens} + 10 \text{ units}) \times 10 \\
 \hline
 31 \ 25 \\
 \hline
 31 \ 25 = (2 \cdot 31 \text{ tens} + 5 \text{ units}) \times 5
 \end{array}
 \end{array}$$

Therefore ± 315 are the square roots of 99,225.

When the method and reasons for the process have become familiar, the work may be shortened by omitting the explanations and unnecessary zeros as follows:

$$\begin{array}{r}
 \begin{array}{r}
 \underline{3 \ 6} \\
 12 \ 96 \\
 \hline
 9 \\
 66 \overline{) 3 \ 96} \\
 \underline{3 \ 96}
 \end{array}
 \qquad
 \begin{array}{r}
 \underline{1 \ 1 \ 6} \\
 \widehat{1 \ 34 \ 56} \\
 1 \\
 21 \overline{) 34} \\
 \underline{21} \\
 226 \overline{) 13 \ 56} \\
 \underline{13 \ 56}
 \end{array}
 \end{array}$$

The foregoing method is the one commonly used for extracting the square root of a number. For it we have the

RULE. *Begin at the decimal point and point off as many groups of two digits each as possible: to the left if the number is an integer; to the right if it is a decimal; to both the left and the right if the number is part integral and part decimal.*

Find the greatest integer whose square is equal to or less than the left-hand group, and write this integer for the first digit of the root and directly over the group of digits used in determining it.

Square the first digit of the root, subtract its square from the first group, and annex the second group to the remainder.

Double the part of the root already found for a trial divisor, divide it into the remainder (omitting from the latter the right-hand digit), and write the integral part of the quotient as the next digit of the root and directly over the group of digits used in determining it.

Annex the root digit just found to the trial divisor to make the complete divisor, multiply the complete divisor by this root digit, subtract the result from the dividend, and annex to the remainder the next group for a new dividend.

Double the part of the root already found for a new trial divisor and proceed as before until the desired number of digits of the root have been found.

After extracting the square root of a number involving decimals, point off one decimal place in the root for every decimal group in the number.

CHECK. *If the root is exact, square it. The result should be the original number. If the root is inexact, square it and add to this result the remainder. The sum should be the original number.*

Sometimes in using a trial divisor we obtain too great a quotient for the next digit of the root. This happens in obtaining the second digit of the square root of 32,301, where 2 into 22 gives 11. Obviously 10 and 11 are both impossible. If 9 is tried we get $9 \cdot 29$, or 261, which is greater than 223. Similarly, 8 is too great. But $7 \cdot 27 = 189$, which is less than 223. Therefore 7 is the second digit of the root.

$$\begin{array}{r} 1 \\ \overline{32\widehat{3}01} \\ 2 \overline{)223} \end{array}$$

Occasionally the trial divisor gives a quotient less than 1. This indicates that the required root digit is 0, which should be written in the root and the work continued as usual. An instance of this kind occurs in finding the second digit in the square root of 942.49. The quotient of $4 \div 6$ is $\frac{2}{3}$, which is not an integer. Therefore the second digit of the root is 0. Then the next period, 49, should be brought down. The new trial divisor will be 60, which will give 7 as the third digit of the root. The work can easily be completed, giving 30.7 as the square root.

$$\begin{array}{r} 30 \\ \hline 942.49 \\ 6 \overline{) 42} \end{array}$$

An attempt to extract the square root of 2 by annexing decimal periods of zeros and applying the rule becomes a never-ending process.

$$\begin{array}{r} 1.414 \\ \hline 2.000000 \\ 24 \overline{) 100} \\ \quad 96 \\ \hline 281 \overline{) 400} \\ \quad 281 \\ \hline 2824 \overline{) 11900} \\ \quad 11296 \end{array}$$

The number 2 has no exact square root, and no matter how far the work is carried, there is no final digit. As the work stands, we know that the square root of 2 lies between 1.414 and 1.415.

116. Significant figures. The digits that are retained as *correct* are called the significant figures. The digit 0 may or may not be significant. In the first computation on this page it was significant because it indicated the correct result in that decimal place.

As a further example consider the number 3600. We may say that 3600 expresses the number 3629 correct to two digits, or to two significant figures. In this case neither of the two 0's is significant. The number 3629 correct to three places is 3630. The number 4692 expressed correct to three significant figures is 4700, the first 0 being significant, the second 0 not significant.

ORAL EXERCISES

Express the following numbers correct to three significant figures :

- | | | | | |
|-----------|----------|------------|---------------|------------|
| 1. 27.61. | 3. 4969. | 5. 3.1416. | 7. 24.96. | 9. 1.414. |
| 2. 3428. | 4. 9064. | 6. 3902. | 8. 5,432,196. | 10. 1.732. |

EXERCISES

Find the square root of the following numbers, correct to the nearest hundredth :

- | | | | |
|----------|------------|-----------|------------------------|
| 1. 24. | 7. 3.1416. | 13. 1.3. | 19. 4,563,210. |
| 2. 8. | 9. 549. | 14. .25. | 20. 87,616. |
| 3. 1.53. | 9. 10000. | 15. 1924. | 21. 907,853. |
| 4. 120. | 10. 125. | 16. 5160. | 22. 25,200. |
| 5. 2. | 11. 1563. | 17. 2395. | 23. $\sqrt{301,957}$. |
| 6. 7. | 12. 2000. | 18. 8753. | 24. 2,156,873. |

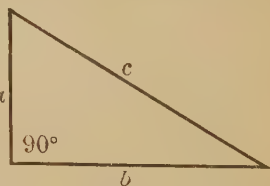
Fact from Geometry. In the adjacent right triangle, $a^2 + b^2 = c^2$, the sides a and b , which form the right angle, are called the *sides*; and c , the side opposite the right angle, is called the *hypotenuse*.

If side a is 8 and side b is 15, $a^2 + b^2 = c^2$ then substituting in $a^2 + b^2 = c^2$ gives $64 + 225 = c^2$. Whence $289 = c^2$ and $c = \pm 17$.

Since -17 is not a practical answer, it is rejected.

In Exercises 25-28 find the hypotenuse and the area of a right triangle whose sides are :

- | | |
|------------------------------|--------------------------------|
| 25. 3 feet and 4 feet. | 27. 125 inches and 180 inches. |
| 26. 16 inches and 12 inches. | 28. 293 inches and 873 inches. |



In Exercises 29–32 find the hypotenuse and side of a right triangle whose area and other side are respectively :

29. 2 square feet and 1 foot.
30. 45 square inches and 15 inches.
31. 132 square feet and 12 feet.
32. 21 square inches and 3 inches.

In Exercises 33–35 find the other side of a right triangle in which the hypotenuse and one side are respectively :

33. 213 feet and 25 feet.
34. 8.5 miles and .29 mile.
35. 3.75 yards and .67 yard.

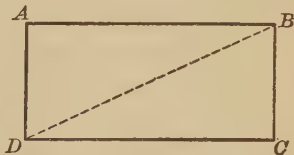
Extract the square root in Exercises 36–39 inclusive to the nearest ten thousandth, and in Exercises 40–44 inclusive to the nearest thousandth.

A fractional number should be reduced to a decimal before extracting the square root, unless the root is seen to be exact.

- | | | |
|---------------|--------------|-----------------------|
| 36. 3.14159. | 39. .001432. | 42. $\frac{11}{32}$. |
| 37. 2.71828. | 40. 5. | 43. $2\frac{7}{8}$. |
| 38. 1034.266. | 41. 7. | 44. $\frac{17}{64}$. |

In the following find all roots to the nearest hundredth.

In rectangle $ABCD$ line DB is called a *diagonal*.



45. Find the diagonal of a rectangle whose sides are 7 inches and 18 inches respectively.

46. The diagonal of a rectangle is 25 feet. Its perimeter is 70 feet. What are its length and its breadth?

47. One side of a rectangle is 10 inches. The diagonal is 15 inches. Find the perimeter and the area of the figure.

48. The side of a square is 25 inches. Find the diagonal.

49. The length of a rectangular figure is 23 feet. If the diagonal is twice the shorter side, what is the width?

50. The diagonal of a rectangle is 37 feet. The length is $1\frac{1}{2}$ times the width. Find the length and the breadth of the figure, and its area.

51. The diagonal of a rectangular figure is 25 feet. If the two sides are equal in length, what is the side? the area?

52. The distance traveled by a freely falling body is expressed by the formula $s = 16t^2$. In this expression s is the distance traveled in feet, while t is the time of fall in seconds. How long will it take a body to fall 100 feet? 40 feet?

53. The diagonal of a rectangle is twice the shorter side. The area of the figure is 580.378 square feet. Find the length, the breadth, and the diagonal of the figure.

54. The width of a rectangle is 15 feet less than the length. The diagonal is 75 feet. Find the dimensions and the area.

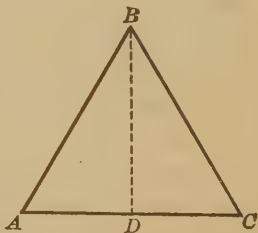
55. The area of a circle is expressed by the equation $A = \pi r^2$. What is the radius (r) of a circle whose area (A) is 75 square feet? 123 square feet? ($\pi = 3.14$.)

Fact from Geometry. A line drawn from one vertex of an equilateral triangle to the middle point of the opposite side is perpendicular to that side.

Thus, in the equilateral triangle ABC , if D is the middle point of AC , BD is the altitude; and

$$\overline{BD}^2 = \overline{AB}^2 - \overline{AD}^2 = \overline{AB}^2 - \left(\frac{AC}{2}\right)^2.$$

56. The perimeter of an equilateral triangle is 18 inches. What is the area?



57. In the figure of the triangle given above, what is the ratio of the altitude to the side of the triangle?

58. Using the ratio obtained above, find the side of an equilateral triangle whose area is 118 square feet.

59. The formula for the amount (A) which a sum of money (P) amounts to at compound interest for a term of years (n) is $A = P(1 + r)^n$ where r is the rate of interest expressed as a decimal fraction. What must be the interest rate to make \$75 grow to \$80 in 2 years? in 4 years?

60. A projectile fired horizontally from a gun will strike the level ground in a length of time the same as that for a body falling from rest the distance from the muzzle of the gun to the ground. The expression for this is $s = 16 t^2$, where s is the distance in feet of the barrel of the gun above the level plain and t is the time in seconds. How far would a projectile travel if its average velocity were 1000 feet per second and the muzzle of the gun were 5 feet above the ground?

(The solution of the above problem neglects air resistance and the result obtained will therefore be only approximate.)

NOTE. A method of extracting the square root of numbers not unlike that in use today was employed by a Greek, Theon, about A. D. 350. In the Middle Ages square roots were extracted with a fair degree of accuracy by using the formulas of approximation:

$$(1) \sqrt{a^2 + x} = a + \frac{x}{2a} \quad (2) \sqrt{a^2 + x} = a + \frac{x}{2a + 1}$$

The true value of the square root of the number was proved to be between the results obtained by these expressions. Thus, if $\sqrt{65}$ was desired, it was noticed that $65 = 64 + 1$, and from (1)

$$\sqrt{65} = \sqrt{64 + 1} = \sqrt{8^2 + 1} = 8 + \frac{1}{2 \cdot 8} = 8\frac{1}{16},$$

while from (2) $\sqrt{65} = \sqrt{64 + 1} = \sqrt{8^2 + 1} = 8 + \frac{1}{2 \cdot 8 + 1} = 8\frac{1}{17}.$

Thus the true value of $\sqrt{65}$ is between these two numbers. This method was known to the Arabs.

It should be kept in mind that the use of decimal fractions and of the decimal point was not common until the eighteenth century. Consequently the complete development of the method of extracting the square root given in the text is comparatively recent.

REVIEW EXERCISES

Find the square root of the following numbers correct to four significant figures:

- | | | |
|---------------|----------------|------------------|
| 1. 3.1459. | 4. 37,111,519. | 7. 87.5476351. |
| 2. 149,253. | 5. 2,502.598. | 8. 28,753.24935. |
| 3. 2,150,000. | 6. 34,875,412. | 9. 3.587.281. |

Factor the following expressions:

- | | |
|----------------------------|-----------------------------|
| 10. $10a^2 + ab - 3b^2$. | 13. $1 + 24m^2 + 144m^4$. |
| 11. $10x^2 - x - 21$. | 14. $169 - x^2$. |
| 12. $12x^2 - 112x - 147$. | 15. $12x^2 - 11ax - 5a^2$. |

16. The horse power transmitted by a leather belt in a shop is expressed by the formula $HP = \frac{(T - t)V}{33,000}$. T and t are the pulls of the belt on each side of the pulley, and V is the speed of the belt in feet per minute. What horse power is transmitted by a belt having pulls of 200 and 150 pounds respectively and a speed of 3000 feet per minute?

17. An automobile has a wheel 32 inches in diameter. How many revolutions per minute is the wheel making when the car is traveling 25 miles per hour? 30 miles per hour?

18. Two trains running between the same two stations take 20 minutes and 30 minutes respectively. One train travels 15 miles per hour faster than the other. What are their speeds?

19. A party contains 45 people. If there were 9 more women there would be half as many women as men. How many men and women are there in the party?

20. A man travels 25 miles, riding part of the way and walking the rest. If he walked 7 miles more than he rode, how far did he ride?

21. How much rope will be needed to reach from the top of a tent pole 10 feet high to a point on the ground 7 feet from the base of the pole?

CHAPTER XXI

RADICALS

117. Rational numbers. The quotient of two integers is called a *rational number*.

Any integer is a rational number, since it may be considered as the quotient of itself and 1.

Thus 4, -2 , $1\frac{1}{3}$, $-\frac{8}{9}$, and 3.1416 are rational numbers. All fractions and decimal fractions are also rational numbers.

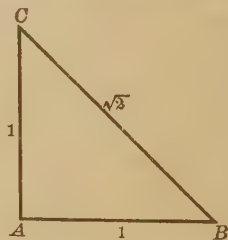
118. Radical. A *radical* is an indicated root of an algebraic or arithmetic expression.

Thus $\sqrt{19}$, $\sqrt{2}$, $\sqrt[4]{a}$, and $\sqrt{x^2 - 3x + 2}$ are radicals.

If a number under a radical sign is such that the root cannot be obtained exactly, the radical represents an *irrational number*.

Thus $\sqrt{2}$, $\sqrt[3]{2}$, $\sqrt{3}$, $\sqrt[3]{5}$, are irrational numbers, since the indicated roots of 2, of 3, and of 5 will never come out even however far the process of extracting the root is carried.

Though no irrational numbers can be expressed exactly in decimals, we can represent some of them by the lengths of lines. Thus, in the right triangle ABC , if $AB = AC = 1$ inch, $BC = \sqrt{2}$ inches. If $AB = 2$ inches and $AC = 1$ inch, $BC = \sqrt{5}$ inches.



There are other types of irrational numbers which cannot be expressed in terms of radicals, but their consideration is too complicated for this text.

119. Imaginaries. If a number under a square-root sign is negative, the radical represents an *imaginary* number.

Thus $\sqrt{-2}$, $\sqrt{-8}$, and $\sqrt{-4}$ are imaginary numbers.

If the student pursues the study of algebra further he will learn that imaginary numbers are required to express completely the cube and higher roots of any positive or negative number.

For example, he will learn that the number 27 has *two other* cube roots besides the number 3. Imaginary numbers, however, are not considered in this text.

120. Index. The small figure, like the 3 in $\sqrt[3]{m}$, is called the *index* of the radical. 2

In $3\sqrt[4]{2}$, 4 is the index.

121. Radicand. The *radicand* is the number or expression under the radical sign.

In $\sqrt{3}$ and $\sqrt[3]{xy}$, 3 and xy are the radicands.

122. Principal root. If a number has two roots for a given index, the positive one is called the *principal root*.

When no plus or minus sign precedes the radical sign, the principal root is always implied.

For example, the number 4 has two square roots, + 2 and - 2. Here + 2 is the principal square root of 4, and the radical $\sqrt{4}$ means + 2 and not - 2. Thus $\sqrt{4} = 2$, but $-\sqrt{4} = -2$.

In case a number has only one real root for a given index, that root is the principal root.

For example, the principal root of $\sqrt[3]{-8}$ is - 2; that of $\sqrt[3]{-27}$ is - 3.

ORAL EXERCISES

In the following tell which number is the index and which is the radicand :

1. $\sqrt[3]{4}$. 2. $\sqrt[4]{16}$. 3. $\sqrt[5]{10}$. 4. $\sqrt{2}$. 5. $\sqrt[12]{13}$. 6. $\sqrt[10]{3}$.

Which of the following are rational numbers?

7. $\sqrt{4}$. 9. $\sqrt{12}$. 11. $\sqrt{3}$. 13. $\sqrt[3]{9}$. 15. $\sqrt[4]{1}$.
 8. $\sqrt[3]{8}$. 10. $\sqrt[4]{81}$. 12. $\sqrt{2}$. 14. $\sqrt{8}$. 16. $\sqrt[3]{125}$.

What is the principal square root of the following?

17. 9. 18. 36. 19. 81.

What is the principal cube root of the following?

20. 8. 21. - 27. 22. 125. 23. - 125.

What is the principal fifth root of the following?

24. 1. 25. - 1. 26. 32. 27. - 32. 28. - 243.

123. Simplification of radicals. The form of a radical expression may be changed without altering its numeric value. Such changes are necessary for many reasons. For example, the numeric value of a radical expression is most easily obtained from its simplest form. It will be made clear later that $\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$. Granting that the two

fractions are really equal, one can see that the value to several decimal places can be computed more easily from the second fraction than from the first, since fewer long numeric operations are then necessary.

EXAMPLES

Study the following changes of form :

1. $\sqrt{8} = \sqrt{4 \cdot 2} = \sqrt{4} \cdot \sqrt{2} = 2\sqrt{2}.$
2. In general, $\sqrt{a^2b} = \sqrt{a^2} \sqrt{b} = a\sqrt{b}.$
3. Also, $\sqrt[3]{24} = \sqrt[3]{8 \cdot 3} = \sqrt[3]{8} \sqrt[3]{3} = 2\sqrt[3]{3}.$
4. In general, $\sqrt[3]{a^3b} = \sqrt[3]{a^3} \sqrt[3]{b} = a\sqrt[3]{b}.$

The preceding examples 1 and 2 illustrate the following rule for simplifying a radical involving the square root of an integer or an integral expression.

RULE. Separate the radicand into two factors, one of which is the greatest perfect square which the radicand contains. Then take the square root of this factor and write it as the coefficient of a radical of which the other factor is the radicand.

If the radical already has a coefficient other than the number 1, multiply the result obtained above by this coefficient.

A similar rule holds for radicals involving the cube and higher roots.

Thus $\sqrt[3]{16} = \sqrt[3]{8 \cdot 2} = 2\sqrt[3]{2},$

and $\sqrt[5]{96} = \sqrt[5]{32 \cdot 3} = 2\sqrt[5]{3}.$

NOTE. Although the Arabs were by no means able to state all the rules explained in this chapter, it is interesting to note that they did recognize the truth of a few of them. For instance, a writer about A.D. 830 gives, in his own notation of course, the facts contained in the formulas $\sqrt{a^2b} = a\sqrt{b}$ and $\sqrt{ab} = \sqrt{a}\sqrt{b}.$

EXERCISES

Simplify :

- | | | | |
|-----------------|------------------|------------------|-------------------|
| 1. $\sqrt{12}.$ | 4. $\sqrt{18}.$ | 7. $5\sqrt{20}.$ | 10. $3\sqrt{45}.$ |
| 2. $\sqrt{27}.$ | 5. $\sqrt{125}.$ | 8. $\sqrt{32}.$ | 11. $2\sqrt{98}.$ |
| 3. $\sqrt{75}.$ | 6. $\sqrt{162}.$ | 9. $\sqrt{48}.$ | 12. $5\sqrt{12}.$ |

13. $\sqrt{147}$. 17. $2\sqrt[3]{128}$. 21. $\sqrt{4x}$. 25. $\sqrt{16a}$.
 14. $\sqrt[3]{16}$. 18. $3\sqrt[3]{40}$. 22. $\sqrt{3x^2}$. 26. $\sqrt{4x^2}$.
 15. $\sqrt[3]{24}$. 19. $5\sqrt[3]{96}$. 23. $\sqrt{12m}$. 27. $3\sqrt{9x}$.
 16. $\sqrt[3]{54}$. 20. $\sqrt[3]{81}$. 24. $\sqrt{5r^2}$. 28. $5\sqrt{5m^2}$.
 29. $12\sqrt{8s^2}$. 34. $\sqrt[3]{5m^3}$. 40. $12\sqrt{8x^4}$.
 30. $\sqrt{x^3}$. 35. $\sqrt[3]{3t^3}$. 41. $\sqrt{(x+y)^2a}$.
 HINT. $\sqrt{x^3} = \sqrt{x^2 \cdot x}$. 36. $\sqrt[3]{t^4}$. 42. $\sqrt{2x(b+c)^2}$.
 31. $7\sqrt{4x^3}$. 37. $2\sqrt[3]{8m}$. 43. $\sqrt{4(s+t)^2x}$.
 32. $\sqrt[3]{2x^3}$. 38. $4\sqrt[3]{2x^3}$. 44. $\sqrt{r(x-y)^2}$.
 33. $\sqrt[3]{8x}$. 39. $3\sqrt[3]{54x^3}$. 45. $3\sqrt{(x+z)^2 4a}$.
 46. $(a-b)\sqrt{(a+b)^2c}$. 50. $\sqrt{x^2 + 4x + 4}$.
 47. $\sqrt{(a+b)(a^2-b^2)}$. 51. $\sqrt{2x^2 - 8x + 8}$.
 48. $\sqrt{(x-y)(x^2-y^2)}$. 52. $\sqrt{3x^2 - 30x + 75}$.
 49. $\sqrt{(r+2)(r^2-4)}$. 53. $\sqrt{5r^2 + 5 - 10r}$.

124. Addition and subtraction of radicals. Radicals having the same index and identical radicands are really similar terms. They can be added or subtracted and the result expressed as one term according to the rule on page 48.

Thus $2\sqrt{3} - 4\sqrt{3} + 5\sqrt{3} = 3\sqrt{3}$.

If the radicands are not identical and cannot be simplified further, the radicals are really dissimilar terms, and addition or subtraction can then only be indicated (see page 50).

Thus $\sqrt{5}$, $\sqrt{3}$, and $\sqrt[3]{6}$ are three dissimilar radicals, and the addition of the three can only be indicated thus: $\sqrt{5} + \sqrt{3} + \sqrt[3]{6}$.

ORAL EXERCISES

Simplify :

- | | |
|--|---|
| 1. $2\sqrt{2} + \sqrt{2}$. | 9. $\sqrt{20} + 2\sqrt{5}$. |
| 2. $5\sqrt{3} - 2\sqrt{3}$. | 10. $3\sqrt{8} + 3\sqrt{2} - 5\sqrt{32}$. |
| 3. $4\sqrt{2} + 3\sqrt{2}$. | 11. $6\sqrt{5} + 10\sqrt{20} - 3\sqrt{125}$. |
| 4. $12\sqrt{5} - \sqrt{5}$. | 12. $3\sqrt{x} + 2\sqrt{x}$. |
| 5. $3\sqrt{6} - 2\sqrt{6}$. | 13. $2\sqrt{x} + 5\sqrt{4x}$. |
| 6. $4\sqrt{3} + 5\sqrt{3} - 6\sqrt{3}$. | 14. $3\sqrt{16a} + 5\sqrt{4a}$. |
| 7. $4\sqrt{7} - 2\sqrt{7} + 3\sqrt{7}$. | 15. $\sqrt{b} + \sqrt{a^2b}$. |
| 8. $7\sqrt{3} + 4\sqrt{12}$. | 16. $a\sqrt{75} - \sqrt{48a^2}$. |

EXERCISES

Simplify and collect :

- | | |
|---|---|
| 1. $\sqrt{8} + \sqrt{2} + 2\sqrt{32}$. | 12. $\sqrt{a^3} + a\sqrt{a} + \frac{1}{a}\sqrt{a^5}$. |
| 2. $\sqrt{12} + \sqrt{27}$. | 13. $\sqrt{a^2b} + \sqrt{a^2b^3} + \sqrt{b}$. |
| 3. $\sqrt{3} + \sqrt{48}$. | 14. $3\sqrt{x} + 4\sqrt{9x} + 5\sqrt{4x}$. |
| 4. $2\sqrt{32} - 5\sqrt{18}$. | 15. $a\sqrt{\frac{x}{a^2}} + 3\sqrt{x} + 2\sqrt{16x}$. |
| 5. $3\sqrt{40} + \sqrt{250}$. | 16. $x + 3\sqrt{3x^2}$. |
| 6. $4\sqrt{27} + 2\sqrt{147}$. | 17. $\sqrt{a^3} + \sqrt{b^2a} + \sqrt{4a}$. |
| 7. $\sqrt{8} + \sqrt{32} - \sqrt{50}$. | 18. $2\sqrt{x} - 3\sqrt{4x} + \frac{1}{x}\sqrt{144x^3}$. |
| 8. $15\sqrt{75} - \sqrt{27} + 6\sqrt{12}$. | 19. $\sqrt{a^3b^2c} - a\sqrt{ab^2c^3} + a\sqrt{ac}$. |
| 9. $3\sqrt{27} - 5\sqrt{3}$. | 20. $4\sqrt{3a^3} + \sqrt{12a} - 2\sqrt{27a^5}$. |
| 10. $\sqrt{5} + 3\sqrt{125}$. | |
| 11. $3\sqrt{x^3} + 5x\sqrt{x}$. | |

In the following exercises find the value of the expression, first by extracting the square root of each term and

adding the results; then by simplifying and collecting before extracting the square root. Compare the amount of labor involved in the two methods.

$$21. 3\sqrt{50} + 2\sqrt{2} - 12\sqrt{98}. \quad 22. 4\sqrt{27} - 5\sqrt{75} + 2\sqrt{3}.$$

$$23. \sqrt{125} + 2\sqrt{80} - 3\sqrt{20} + 5\sqrt{45}.$$

125. Simplification of fractional radicands. Radicals whose radicands are fractions or fractional expressions occur frequently in practice, especially in certain parts of geometry.

EXAMPLES

Study the following simplification of fractional radicands:

$$1. \sqrt{\frac{2}{3}} = \sqrt{\frac{6}{9}} = \sqrt{\frac{1}{9} \cdot 6} = \sqrt{\frac{1}{9}} \sqrt{6} = \frac{1}{3} \sqrt{6}.$$

$$2. 6\sqrt{\frac{2}{5}} = 6\sqrt{\frac{10}{25}} = 6\sqrt{\frac{1}{25} \cdot 10} = 6 \cdot \frac{1}{5} \sqrt{10} = \frac{6}{5} \sqrt{10}.$$

$$3. \sqrt{\frac{1}{2x}} = \sqrt{\frac{2x}{4x^2}} = \sqrt{\frac{1}{4x^2} \cdot 2x} = \frac{1}{2x} \sqrt{2x}.$$

$$4. \sqrt{\frac{a}{b}} = \sqrt{\frac{ab}{b^2}} = \sqrt{\frac{1}{b^2} \cdot ab} = \frac{\sqrt{ab}}{b}.$$

These examples illustrate the following rule for simplifying an indicated square root which has a *fractional* radicand.

RULE. *Multiply the numerator and the denominator of the radicand by the least whole number or the simplest expression which will make the resulting denominator a perfect square.*

Then separate the radicand into two factors, one of which is a fraction and at the same time the greatest perfect square which the radicand contains.

Take the square root of this factor and write it as the coefficient of the radical of which the other factor is the radicand. If the original radical has a coefficient, multiply the result as obtained above by this coefficient.

ORAL EXERCISES

Simplify:

1. $\sqrt{\frac{1}{2}}$
2. $\sqrt{\frac{2}{7}}$
3. $\sqrt{\frac{1}{5}}$
4. $\sqrt{\frac{1}{a}}$
5. $3\sqrt{\frac{1}{3}}$
6. $4\sqrt{\frac{3}{4}}$
7. $\sqrt{\frac{x}{y}}$
8. $\sqrt{\frac{2x}{a}}$
9. $a\sqrt{\frac{b^2}{a}}$
10. $\sqrt{\frac{3}{5}}$

EXERCISES

Simplify the following:

1. $\sqrt{\frac{5}{16}}$
2. $\sqrt{\frac{3}{2}}$
3. $\sqrt{\frac{2}{3}}$
4. $\sqrt{\frac{7}{8}}$
5. $\sqrt{\frac{4}{5}}$
6. $\sqrt{\frac{5}{3}}$
7. $\sqrt{\frac{3}{8}}$
8. $\sqrt{\frac{13}{21}}$
9. $2\sqrt{\frac{5}{16}}$
10. $3\sqrt{\frac{2}{3}}$
11. $5\sqrt{\frac{7}{2}}$
12. $2\sqrt{\frac{16}{9}}$
13. $5\sqrt{\frac{9}{5}}$
14. $3\sqrt{\frac{8}{15}}$
15. $7\sqrt{\frac{5}{8}}$
16. $\sqrt{\frac{6m^3}{10}}$
17. $m\sqrt{\frac{21b}{m}}$
18. $5\sqrt{\frac{b^2a}{8}}$
19. $\sqrt{\frac{4}{5}} + \sqrt{\frac{1}{5}}$
20. $3\sqrt{\frac{5}{3}} - 2\sqrt{60}$
21. $4\sqrt{\frac{1}{3}} + 2\sqrt{\frac{75}{4}}$
22. $\sqrt{4} - 3\sqrt[3]{8} + 5\sqrt{9}$
23. $2\sqrt[3]{27} + 5\sqrt{\frac{25}{4}} - 3\sqrt{225}$
24. $4\sqrt{2} + 13\sqrt{8} - 5\sqrt{162}$
25. $15\sqrt[3]{16} + 5\sqrt[3]{54} - 3\sqrt[3]{250}$
26. $\sqrt{16} + \sqrt[3]{64} + \sqrt[4]{256}$
27. $\frac{3\sqrt{3} + 4\sqrt{27} + 15\sqrt{75}}{\sqrt{48}}$
28. $3\sqrt{8} + 5\sqrt{\frac{1}{2}} - 6\sqrt{2} + 7\sqrt{\frac{64}{8}}$
29. $5\sqrt{12} + 2\sqrt{27} - 4\sqrt{3} + 3\sqrt{\frac{1}{3}}$
30. $\sqrt{\frac{16}{7}} + 3\sqrt{28} - 5\sqrt{\frac{4}{7}}$
31. $3\sqrt{4\frac{1}{6}} + 2\sqrt{24} - 9\sqrt{\frac{2}{3}}$
32. $2\sqrt{5} - 3\sqrt{\frac{1}{5}} + 6\sqrt{125} - 6\sqrt{\frac{36}{5}}$
33. $3\sqrt{2} + 5\sqrt{4} - 2\frac{1}{2}\sqrt{18} + 8\sqrt{\frac{1}{8}}$

$$34. 5\sqrt{x^2y + 2xy + y} - 2\sqrt{(x+1)^2y}.$$

$$35. 3\sqrt{4x^3 + 4x^2 + x} + 4\sqrt{(x+1)^2x}.$$

$$36. 12\sqrt{ax^2 - 2ax + a} - 3\sqrt{(x-1)^2a^3}.$$

The need for simplifying radicals presents itself in various problems, as, for example, in geometric work on right triangles.

PROBLEMS

(Obtain answers in *simplest radical form*.)

1. One side of a right triangle is 6 and the other side is 9. Find the hypotenuse.

Solution. $h = \sqrt{36 + 81} = \sqrt{117} = 3\sqrt{13}.$

2. The hypotenuse of a right triangle is 15 and one side is 10. What is the other side? What is the area?

3. The side of a square is 3. What is the diagonal?

4. The diagonal of a rectangle is D , and the width is $\frac{D}{3}$. Find the length and the area.

5. Find the area of a rectangle, of diagonal D , if the length is twice the breadth.

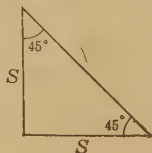
6. Find the sides of a square if the diagonal is 10.

7. What is the side of a square whose diagonal is $3L$?

Problems involving the following classes of triangles are of frequent occurrence in practical work and often require the use of radicals:

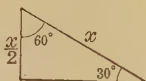
(a) An isosceles right triangle; that is, a right triangle with two equal sides.

As indicated in the figure, if each side is S the two acute angles are 45° each.



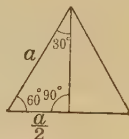
(b) A right triangle with one angle 30° or 60° .

As indicated in the figure, if one acute angle is 30° the other is 60° , and vice versa. More important still, the *hypotenuse* is always *twice the shorter side*.



(c) An equilateral triangle.

As indicated in the figure, when the altitude is drawn, it divides the base into two equal parts, and in each of the right triangles formed the same relations exist as in (b) above.



8. If each of the equal sides of an isosceles right triangle is 4, what is the hypotenuse?

HINT.

$$h^2 = 16 + 16.$$

$$h = \sqrt{16 + 16}, \text{ etc.}$$

9. The hypotenuse of an isosceles right triangle is 15. What is one side?

10. The formula for the area of a circle in terms of its radius is $A = \pi r^2$. What is the radius in terms of the area?

11. What is the radius of a circle whose area is 24 square inches?

12. The formula $C = \frac{5}{9} (F - 32)$ expresses the temperature by the centigrade scale in terms of the readings on the Fahrenheit scale. Solve this equation for F . What is the centigrade temperature corresponding to a Fahrenheit reading of 100° ? 20° ?

13. The area of the surface of a sphere in terms of the radius is $A = 4\pi r^2$. Find the radius in terms of the area.

14. What is the radius of a sphere whose surface is 18 square feet?

15. The volume of a cylinder in terms of its radius and height is $V = \pi r^2 h$. What is the height in terms of V and r ? Find r in terms of h and V .

16. What must be the radius of a cylinder whose volume is 50 cubic inches and whose height is 20 inches?

17. A runway has an inclination of 30° to the horizontal. How far above the ground is a man who has walked along the runway a distance of 40 feet with respect to the horizontal?

HINT. See diagram in (b) on page 311.

18. A ladder makes an angle of 60° with the ground. A man climbs 15 feet up along the ladder. How far is he above the ground?

19. A man desires to make a bank of earth against a perpendicular wall 10 feet high. If the angle of repose of the earth is 45° , how many cubic feet of earth will he need to make a bank 20 feet long? How many square feet of sod will be required to cover the face of the bank?

HINT. The volume of earth required is equal to the length of the bank times the area of cross section, which is in this case an isosceles right triangle of 10-foot side.

20. What is the hypotenuse of an isosceles right triangle whose area is 50 square feet? 25 square feet?

21. The side of an equilateral triangle is 3 feet. Find the altitude and area.

22. The area of an equilateral triangle is $100\sqrt{3}$ square inches. What is the length of a side?

23. A hardware dealer desires to pack triangular files in a box. The files are $\frac{4}{10}$ of an inch on a side. What should be the depth of a box which will just contain one layer of files?

24. The hypotenuse of a right triangle with a 30° angle is 10 feet. What is the area?

NOTE. Though methods of classifying irrational expressions are found in the work of Euclid, the Hindus and the Arabs were the first to develop this part of algebra in a form similar to that used today.

126. Multiplication of radicals. Monomial radicals having the same index are multiplied as shown in the following:

EXAMPLES

1. Multiply $4\sqrt{2}$ by $3\sqrt{6}$.

Solution. $4\sqrt{2} \cdot 3\sqrt{6} = 12\sqrt{12} = 12 \cdot 2\sqrt{3} = 24\sqrt{3}$.

2. Multiply $3\sqrt[3]{2mr^2}$ by $2\sqrt[3]{5m^2r^2}$.

Solution. $3\sqrt[3]{2mr^2} \cdot 2\sqrt[3]{5m^2r^2} = 6\sqrt[3]{10m^3r^4} = 6mr\sqrt[3]{10r}$.

The method just illustrated of multiplying monomial radicals of the same index is stated in the

RULE. *Write the product of the coefficients of the radicals for the coefficient of the radical in the result.*

Multiply together the radicands and write the product under the common radical sign.

Reduce the result to its simplest form.

The preceding rule does not hold for radicals having different indices.

ORAL EXERCISES

Perform the indicated multiplication :

- | | | |
|--------------------------------|----------------------------------|--|
| 1. $\sqrt{2} \cdot \sqrt{3}$. | 4. $\sqrt{3} \cdot \sqrt{6}$. | 7. $3\sqrt{2} \cdot 2\sqrt{5} \cdot 2\sqrt{3}$. |
| 2. $\sqrt{2} \cdot \sqrt{2}$. | 5. $3\sqrt{2} \cdot 2\sqrt{3}$. | 8. $5\sqrt{3} \cdot 4\sqrt{5}$. |
| 3. $\sqrt{2} \cdot \sqrt{5}$. | 6. $5\sqrt{3} \cdot 2\sqrt{7}$. | 9. $4\sqrt{13} \cdot 2\sqrt{2}$. |

EXERCISES

Perform the indicated operation and simplify results?

1. $\sqrt{5} \cdot \sqrt{6}$.

2. $\sqrt{8} \cdot \sqrt{3}$.

3. $2\sqrt{15} \cdot 5\sqrt{12}$.

4. $3\sqrt{3} \cdot 2\sqrt{30}$.

5. $4\sqrt[3]{16} \cdot 10\sqrt[3]{2}$.

6. $6\sqrt{21} \cdot 5\sqrt{7}$.

7. $\sqrt{8} \cdot 4\sqrt{7}$.

8. $5\sqrt{18} \cdot \sqrt{3}$.

9. $m\sqrt{10} \cdot n\sqrt{5}$.

10. $\sqrt{x} \cdot b^2\sqrt{y}$.

11. $\sqrt{\frac{25a}{b}} \cdot \sqrt{\frac{4a^3}{b}}$.

12. $\sqrt{x^3} \cdot 5\sqrt{\frac{3}{x}}$.

13. $10\sqrt{m} \cdot 5\sqrt{\frac{1}{m}}$.

14. $5\sqrt{x} \cdot 5\sqrt{x}$.

15. $\sqrt{\frac{2m}{b}} \cdot \sqrt{\frac{2b}{m}}$

16. $\sqrt{\frac{6x^3}{n}} \cdot \sqrt{\frac{n^3}{5x}}$.

17. $\sqrt{\frac{2m}{3}} \cdot \sqrt{\frac{3m}{2}}$.

18. $12\sqrt{\frac{x^3}{5}} \cdot 4\sqrt{\frac{x}{4}}$.

19. $2\sqrt{\frac{x^2}{y}} \cdot x\sqrt{\frac{y^3}{2}}$.

20. $\sqrt{2}(\sqrt{2} + 1)$.

HINT. $\sqrt{2}(\sqrt{2} + 1) = \sqrt{4} + \sqrt{2}$, etc.

21. $\sqrt{3}(\sqrt{3} - \sqrt{2})$.

22. $\sqrt{2}(2\sqrt{2} - \sqrt{5})$.

23. $\sqrt{5}(\sqrt{3} - 2\sqrt{2})$.

24. $2\sqrt{3}(\sqrt{3} - \sqrt{2})$.

25. $3\sqrt{5}(\sqrt{2} - \sqrt{3} + \sqrt{5})$.

26. $(\sqrt{5} - \sqrt{8} - \sqrt{3})(\sqrt{3})$.

27. $(\sqrt{2} - \sqrt{10} + \sqrt{20})(-2\sqrt{5})$.

Find the radical expression having the coefficient 1 which is equivalent to each of the following:

28. $4\sqrt{3}$.

29. $5\sqrt{5}$.

32. $4\sqrt{\frac{1}{2}}$.

Solution. $4\sqrt{3} = \sqrt{16}\sqrt{3}$

30. $3\sqrt{2}$.

33. $3\sqrt{\frac{1}{3}}$.

$= \sqrt{48}$.

31. $10\sqrt{13}$.

34. $5\sqrt{\frac{3}{5}}$.

35. $a\sqrt{x}$.

36. $3b\sqrt{y}$.

37. $4a\sqrt{2z}$.

38. $2x\sqrt{\frac{a}{y}}$.

39. $(x+y)\sqrt{\frac{3}{x+y}}$.

The multiplication of binomial or of polynomial radical expressions of the same order involves no new principle.

EXAMPLE

Multiply $3\sqrt{5} - 4\sqrt{3}$ by $2\sqrt{5} + \sqrt{3}$.

$$\begin{array}{r}
 \text{Solution.} \quad 3\sqrt{5} - 4\sqrt{3} \\
 \quad \quad \quad 2\sqrt{5} + \sqrt{3} \\
 \hline
 30 \quad - 8\sqrt{15} \\
 \quad \quad + 3\sqrt{15} - 12 \\
 \hline
 30 \quad - 5\sqrt{15} - 12 = 18 - 5\sqrt{15}.
 \end{array}$$

EXERCISES

Perform the indicated multiplication and simplify the results:

1. $(\sqrt{2} + 5)(3\sqrt{2} - 4)$.

3. $(6\sqrt{5} - 2)(\sqrt{5} + 3)$.

2. $(7\sqrt{3} + 2)(\sqrt{3} - 5)$.

4. $(4 + 3\sqrt{5})^2$.

5. $(5\sqrt{5} + 2\sqrt{3})(2\sqrt{5} - 3\sqrt{3})$.

6. $(2\sqrt{3} + 5\sqrt{2})(2\sqrt{3} + 2\sqrt{2})$.

7. $(3\sqrt{6} - 2\sqrt{5})(12\sqrt{5} + 13\sqrt{6})$.

8. $(4 + \sqrt{2} - 2\sqrt{3})(\sqrt{5} + \sqrt{2} + 6)$.

9. $(3 - 2\sqrt{3} + \sqrt{2})(3\sqrt{2} + \frac{1}{2}\sqrt{\frac{1}{3}})$.

10. $(\sqrt{10} + \sqrt{15} - \sqrt{18})(\sqrt{2} + \sqrt{60} + 3\sqrt{90})$.

11. $(\sqrt{x} + 3\sqrt{y})(\sqrt{x} - 3\sqrt{y})$.

12. $(4\sqrt{m} + 3\sqrt{n})(3\sqrt{m} + 4\sqrt{n})$.

$$13. (2\sqrt{x} + 3\sqrt{y})(\sqrt{x} - 2\sqrt{y}).$$

$$14. (2\sqrt{3}x - 5\sqrt{2}y)(3\sqrt{12}x + 2\sqrt{2}y).$$

$$15. \sqrt{x^2 - y^2} \cdot \sqrt{3(x + y)}.$$

$$16. \sqrt{(a^2 + 2ab + b^2)^3} \cdot \sqrt{(a^2 - b^2)(a + b)}.$$

$$17. (\sqrt{2} + \sqrt{3} - \sqrt{5})(3 + \sqrt{2}).$$

$$18. (3\sqrt{6} + 2\sqrt{7} + 3\sqrt{14})(4 - \sqrt{7}).$$

$$19. (3\sqrt{x+1} - 4\sqrt{x-1})(\sqrt{x^2-1}).$$

$$20. (2\sqrt{x-a} + 3\sqrt{x-4a})(\sqrt{x-a}).$$

$$21. (\sqrt{m+t} - 2\sqrt{m} + 3\sqrt{m-t})(\sqrt{m}).$$

$$22. (3\sqrt{2+x} - 5\sqrt{x-2} + 13\sqrt{x})(4\sqrt{x+2}).$$

23. The hypotenuse of a right triangle is R , and one side is $\frac{R\sqrt{3}}{2}$. Find the other side.

24. One side of a right triangle is $\sqrt{2}(R\sqrt{2} - 1)$, and the hypotenuse is R . What is the other side?

25. The sides of a right triangle are $\frac{1}{2}x\sqrt{2\sqrt{2}-2}$ and $\frac{1}{2}x\sqrt{2\sqrt{2}+2}$. What is the hypotenuse of the triangle? the area?

26. Show by substituting and simplifying that $2 + 2\sqrt{3}$ is a root of the quadratic equation $x^2 - 4x = 8$.

27. Does $\frac{1}{2}(5 - \sqrt{5})$ satisfy the equation $x^2 + 5 = 5x$?

127. Rationalization and division by radicals. It is frequently necessary to find the approximate value of an expression which involves division by a radical expression. Thus $3 \div \sqrt{6}$, $\sqrt{2} \div \sqrt{7}$, are types which often occur.

To find the approximate value of $3 \div \sqrt{6}$, we may extract the square root of 6 to several decimal places and then

divide 3 by the approximate root obtained. Both these processes are long and one of them is unnecessary. For, writing $3 \div \sqrt{6}$ as a fraction and multiplying both terms by $\sqrt{6}$ gives $\frac{3\sqrt{6}}{6}$. To find the approximate value of this last fraction requires but one long operation.

One radical expression is the *rationalizing factor* for another if the *product* of the two is a *rational number*.

A rationalizing factor of $\sqrt{2}$ is $\sqrt{2}$, since $\sqrt{2} \cdot \sqrt{2} = 2$.

Similarly, $\sqrt{3}$ is a rationalizing factor of $\sqrt{27}$, since

$$\sqrt{3} \cdot \sqrt{27} = \sqrt{81} = 9.$$

ORAL EXERCISES

Determine a rationalizing factor for each of the following expressions and find the product of the given expression and this factor:

- | | | | | |
|------------------|------------------|------------------|-------------------|-------------------|
| 1. $\sqrt{2}$. | 3. $\sqrt{3}$. | 5. $3\sqrt{7}$. | 7. $5\sqrt{24}$. | 9. $\sqrt{14}$. |
| 2. $2\sqrt{5}$. | 4. $\sqrt{10}$. | 6. $\sqrt{48}$. | 8. $\sqrt{8}$. | 10. $2\sqrt{x}$. |

Direct division of similar radicals, coefficient by coefficient and radicand by radicand, is often possible.

Thus $\sqrt{6} \div \sqrt{2} = \sqrt{3}$.

And $12\sqrt{10} \div 2\sqrt{5} = 6\sqrt{2}$.

$$(\sqrt{10} - \sqrt{15}) \div \sqrt{5} = \sqrt{2} - \sqrt{3}.$$

But $\sqrt{7} \div \sqrt{3} = \frac{\sqrt{7}}{\sqrt{3}} = \frac{\sqrt{7} \cdot \sqrt{3}}{\sqrt{3} \cdot \sqrt{3}} = \frac{\sqrt{21}}{3}.$

If direct division cannot be exactly performed, we use the

RULE. Write the dividend over the divisor in the form of a fraction. Then multiply the numerator and denominator of the fraction by the rationalizing factor of the denominator and simplify the resulting fraction.

EXERCISES

Rationalize the denominator of the following expressions :

- | | | |
|---------------------------------------|---|--|
| 1. $\frac{\sqrt{10}}{\sqrt{5}}$. | 9. $\frac{\sqrt{5} - \sqrt{10}}{\sqrt{2}}$. | 17. $\frac{a\sqrt{s}}{c\sqrt{t}}$. |
| 2. $\frac{\sqrt{6}}{\sqrt{2}}$. | 10. $\frac{3\sqrt{2} + 5\sqrt{3}}{2\sqrt{3}}$. | 18. $\frac{12\sqrt{x}}{5\sqrt{3}y}$. |
| 3. $\frac{\sqrt{9}}{\sqrt{3}}$. | 11. $\frac{5\sqrt{15}}{4\sqrt{12}}$. | 19. $\frac{4\sqrt{54}}{3\sqrt{50}}$. |
| 4. $\frac{\sqrt{3}}{\sqrt{2}}$. | 12. $\frac{3}{\sqrt{b}}$. | 20. $\frac{2a\sqrt{x}}{3\sqrt{2}x^3}$. |
| 5. $\frac{3\sqrt{7}}{2\sqrt{3}}$. | 13. $\frac{5}{2\sqrt{x}}$. | 21. $\frac{4\sqrt{mn}}{3m\sqrt{a}}$. |
| 6. $\frac{\sqrt{6}}{\sqrt{12}}$. | 14. $\frac{3y}{21\sqrt{a}}$. | 22. $\frac{12\sqrt{x}}{13a\sqrt{y}}$. |
| 7. $\frac{15\sqrt{24}}{5\sqrt{12}}$. | 15. $\frac{\sqrt{a}}{\sqrt{b}}$. | 23. $\frac{13\sqrt{50}}{10\sqrt{75}}$. |
| 8. $\frac{\sqrt{21}}{5\sqrt{7}}$. | 16. $\frac{2\sqrt{x}}{3\sqrt{2}y}$. | 24. $\frac{3\sqrt{32}}{5\sqrt{12\frac{1}{2}}}$. |

Find to three decimal places the value of :

- | | | |
|----------------------------------|-------------------------------------|--|
| 25. $4 + 5\sqrt{3}$. | 29. $\frac{2}{\sqrt{6}}$. | 31. $\frac{\sqrt{5} - \sqrt{3}}{\sqrt{12}}$. |
| 26. $2 - 6\sqrt{8}$. | | |
| 27. $21\sqrt{3} + 5\sqrt{27}$. | 30. $\frac{5\sqrt{2}}{2\sqrt{5}}$. | 32. $\frac{25\sqrt{2}}{\sqrt{75} - 2\sqrt{3}}$. |
| 28. $13\sqrt{91} + 2\sqrt{13}$. | | |

Express with rational denominators :

- | | | | |
|-----------------------------------|--|---------------------------------------|---------------------------------------|
| 33. $\frac{\sqrt{x}}{\sqrt{y}}$. | 34. $\frac{\sqrt{3b^2a}}{15\sqrt{a^2b}}$. | 35. $\frac{3\sqrt{bx}}{4\sqrt{cx}}$. | 36. $\frac{a\sqrt{n}}{b\sqrt{m^3}}$. |
|-----------------------------------|--|---------------------------------------|---------------------------------------|

128. Fractional exponents. Radical expressions may be written in two ways, with *radical signs* or with *fractional exponents*. The relation between the two will now be explained. To do this it is necessary to extend the meaning of the term *exponent*, which, as defined on page 17, applied to positive integral exponents only. We shall assume that the laws which govern the use of integral exponents hold for fractional exponents also.

The fact that $x^2 \cdot x^3 = x^5$ illustrates the more general law

$$x^a \cdot x^b = x^{a+b},$$

where a and b represent either integers or fractions.

Accordingly, $x^{\frac{1}{2}} \cdot x^{\frac{1}{2}} = x^{\frac{1}{2} + \frac{1}{2}} = x^1$ or x . Since $x^{\frac{1}{2}}$ multiplied by itself gives x , $x^{\frac{1}{2}}$ must be another way of writing $\sqrt{x}$.

Hence $\sqrt{x}$ may be written $x^{\frac{1}{2}}$.

Then $4^{\frac{1}{2}} = \sqrt{4} = 2$, and $(25 a^2)^{\frac{1}{2}} = \sqrt{25 a^2} = 5 a$.

Furthermore, $x^{\frac{1}{3}} \cdot x^{\frac{1}{3}} \cdot x^{\frac{1}{3}} = x^{\frac{1}{3} + \frac{1}{3} + \frac{1}{3}} = x^1 = x$.

Since $x^{\frac{1}{3}}$ is one of the three equal numbers whose product is x , we may say that $x^{\frac{1}{3}}$ is another way of writing $\sqrt[3]{x}$.

Therefore $\sqrt[3]{x}$ may be written $x^{\frac{1}{3}}$.

In general terms, $x^{\frac{1}{n}} = \sqrt[n]{x}$.

Now $x^{\frac{3}{2}} = x^{\frac{1}{2}} \cdot x^{\frac{1}{2}} \cdot x^{\frac{1}{2}} = (x^{\frac{1}{2}})^3 = (\sqrt{x})^3$,
and $x^{\frac{3}{2}} = x^{3 \cdot \frac{1}{2}} = (x^3)^{\frac{1}{2}} = \sqrt{x^3}$.

Hence $x^{\frac{3}{2}} = (\sqrt{x})^3$ or $\sqrt{x^3}$.

In general terms, $x^{\frac{a}{n}} = \sqrt[n]{x^a}$.

Thus $x^{\frac{a}{n}}$ means the n th root of x to the a th power.

The *numerator* of a fractional exponent denotes the *power* to which the radicand is to be raised, while the *denominator*

denotes the *root* to be extracted. Whether one extracts the root first and then raises the result to the power, or vice versa, depends wholly on convenience.

BIOGRAPHICAL NOTE

FRANÇOIS VIETA. The reason that algebra is a universal language lies in the fact that the symbols used to indicate the various operations and relations are widely understood and adopted. This has not always been the case, and for a long time during the early history of the subject there was no accepted notation in algebra, but each man used any symbol that suited him. One of the men who did most to establish a fixed notation was François Vieta (1540–1603), a French lawyer who studied and wrote on mathematics as a pastime. He was in public life during his whole career and was well known for his ability to decipher the hidden meaning of dispatches captured from the enemy.

It was he who established the use of the signs $+$ and $-$ for addition and subtraction, which, to be sure, had been used before his time, but were not generally accepted. He also denoted the known numbers in an equation by the consonants, B, C, D , etc., and the unknowns by the vowels, A, E, I , etc. He recognized the existence of negative roots of equations, but rejected them as absurd.

To denote the second and third powers of the unknown, he used the letters Q (*quadratus*) and C (*cubus*) respectively. Instead of using the sign $=$ he wrote *aeq.* (*aequalis* or *aequatur*). Thus Vieta would have written the equation $x^3 - 8x^2 + 16x = 40$ in the form

$$1 C - 8 Q + 16 N \text{ aeq. } 40.$$

Before the time of Vieta this equation would have been written in a much more primitive notation. For instance, with writers only a little earlier it would appear as

$$\text{Cubus } \overline{m} 8 \text{ Census } \overline{p} 16 \text{ rebus aequatur } 40.$$

Operations on equations in this form would certainly be difficult.

Vieta is further distinguished as being the first man to obtain an exact numerical expression for the number π , which occurs in geometry. His form of expression calls for an infinite number of operations which, of course, could never be performed, but the further one proceeded, the closer would be the approximation obtained. In



François Viète

a certain sense the familiar sign $\sqrt{\quad}$ implies an infinite number of operations, for one can never go through the process of extracting the square root of 2, for instance, and come out even. Vieta's method of denoting π was, however, more involved than this and made use of complicated irrational fractions.

ORAL EXERCISES

Read in radical form :

- | | | | | |
|------------------------|--------------------------|---|--|---------------------------|
| 1. $x^{\frac{1}{2}}$. | 3. $3 b^{\frac{1}{5}}$. | 5. $5 y^{\frac{3}{2}}$. | 7. $3^{\frac{1}{2}} \cdot x^{\frac{3}{2}}$. | 9. $6 m^{\frac{2}{3}}$. |
| 2. $x^{\frac{1}{3}}$. | 4. $2 x^{\frac{2}{3}}$. | 6. $12^{\frac{1}{2}} \cdot y^{\frac{1}{3}}$. | 8. $(12 y)^{\frac{4}{3}}$. | 10. $5 b^{\frac{5}{4}}$. |

Change to fractional exponents :

- | | | |
|----------------------|--------------------------|-------------------------|
| 11. $\sqrt{3}$. | 13. $a^{\sqrt[4]{31}}$. | 15. $3\sqrt{x^3}$. |
| 12. $5\sqrt[3]{x}$. | 14. $\sqrt[3]{a^2}$. | 16. $15\sqrt[5]{m^2}$. |

Give the numerical value of :

- | | | |
|---------------------------|-----------------------------|--------------------------------------|
| 17. $(4)^{\frac{1}{2}}$. | 20. $(4)^{\frac{3}{2}}$. | 23. $(36)^{\frac{1}{2}}$. |
| 18. $(9)^{\frac{1}{2}}$. | 21. $(27)^{\frac{1}{3}}$. | 24. $(-64)^{\frac{2}{3}}$. |
| 19. $(8)^{\frac{1}{3}}$. | 22. $(125)^{\frac{2}{3}}$. | 25. $(\frac{1}{16})^{\frac{3}{2}}$. |

EXERCISES

Change to radical form and simplify results :

- | | | |
|---|--|---------------------------------------|
| 1. $m^{\frac{3}{2}}$. | 8. $4(16x)^{\frac{1}{4}}$. | 15. $(\frac{1}{m})^{\frac{3}{2}}$. |
| 2. $2x^{\frac{1}{2}}y^{\frac{3}{2}}$. | 9. $a^{\frac{2}{5}} \cdot x^{\frac{3}{5}}$. | |
| 3. $12x^{\frac{2}{3}}y^{\frac{15}{3}}$. | 10. $3m^{\frac{3}{2}} \cdot 4n^{\frac{5}{2}}$. | 16. $(\frac{2}{x})^{\frac{5}{2}}$. |
| 4. $5^{\frac{3}{2}} \cdot am^{\frac{4}{3}}$. | 11. $12x^{\frac{3}{2}} \cdot y^{\frac{5}{4}}$. | 17. $(\frac{12}{5y})^{\frac{1}{2}}$. |
| 5. $5(rt)^{\frac{3}{2}}$. | 12. $6a^{\frac{15}{4}} \cdot b^{\frac{12}{5}}$. | |
| 6. $(-8)^{\frac{2}{3}} \cdot x^{\frac{5}{2}}$. | 13. $(16y)^{\frac{3}{2}}$. | 18. $(\frac{2x}{3})^{\frac{3}{2}}$. |
| 7. $c(de)^{\frac{3}{2}}$. | 14. $5m^{\frac{2}{4}}y^{\frac{3}{4}}$. | |

REVIEW EXERCISES

Evaluate, carrying all inexact answers to two decimal places:

$$1. 2\sqrt{bm^2} + 3\sqrt{-b^2mx} - 4\sqrt{-bmx},$$

when $b = 1; x = 2; m = -2.$

$$2. 2\sqrt{x^2 + \sqrt{5}br},$$

when $x = 4; b = \frac{1}{5}; r = 81.$

$$3. 5a\sqrt{7x^2 - ay},$$

when $a = 7.2; x = 3; y = 4.8.$

$$4. 2x\sqrt{\frac{m^2 - x^2}{a}},$$

when $m = 3; x = 2; a = 20.$

$$5. 3m + 2\sqrt{\frac{m}{4} + b},$$

when $m = 1; b = 5.$

$$6. \frac{a}{\sqrt{b}},$$

when $a = 5; b = 7.$

$$7. \frac{2\sqrt{x} + \sqrt{y}}{\sqrt{x+y}},$$

when $x = 2; y = 3.$

$$8. y\sqrt{3r^2 + 2ry + y^2},$$

when $r = 5; y = 2.$

9. Express in a form more convenient for computation:

$$(a) \quad r = \sqrt{\frac{\pi v^3}{2.5}},$$

$$(b) \quad r = \sqrt{\frac{A}{\pi}}.$$

Solve for x and y and check:

$$10. \frac{13x + 4}{2} = 15.$$

$$11. \frac{x - a}{x - b} = \frac{b}{a}.$$

$$12. \frac{1}{ab} + 1 - \frac{ab}{x} = \frac{1}{x}.$$

$$13. x^2 + 5x - 24 = 0.$$

$$14. \begin{cases} 3x + 5y = 13, \\ \frac{2x - 15y}{3} = 9. \end{cases}$$

$$15. \begin{cases} \frac{3}{x} + \frac{4}{y} = 7, \\ \frac{1}{x} - \frac{5}{y} = -4. \end{cases}$$

$$16. 4(x^2 - 2x + 1) - 9 = 0.$$

$$17. \begin{cases} .2x + .5y = -1.1, \\ \frac{.3x - .7y}{2} = 1.65. \end{cases}$$

$$18. 4x^2 + 8x + 4 = 0.$$

$$19. \text{Multiply } 2(5b^2x^3y)^{\frac{1}{2}} \text{ by } (3x^2y)^{\frac{1}{2}}.$$

Without solving the equations determine whether the pairs of values given below are roots of the corresponding equations:

$$20. \text{ Do } x = 2\sqrt{5} \text{ and } y = 3\sqrt{5} \text{ satisfy} \\ x + y = 5\sqrt{5}$$

$$\text{and } 3x^2 - 4xy + y^2 = 65?$$

$$21. \text{ Do } x = 2\sqrt{2} \text{ and } y = 5\sqrt{2} \text{ satisfy}$$

$$\frac{2x + 3y}{19} = \sqrt{2}$$

$$\text{and } x^2 + 5xy - 3y^2 = -40?$$

$$22. \text{ Do } a = \sqrt{3} \text{ and } b = \frac{1}{3}\sqrt{3} \text{ satisfy}$$

$$2a - b = \frac{5}{\sqrt{3}}$$

$$\text{and } \frac{a^2 - b^2}{4} = \frac{2}{3}?$$

23. A tree on the bank of a stream is 25 feet from an observer on the opposite bank. The distance from the observer to the top of the tree is 32 feet. If the observer is on a level with the base of the tree, how high is the tree?

24. The altitude of an equilateral triangle is 1 foot. What is the length of one side? What is the area?

25. A man has \$200 invested, part at 4% per year and the remainder at 6% per year. The total income from the money is \$9 per year. How much money is invested at each rate of interest?

26. The sum of three consecutive numbers is 18. Find the numbers.

27. The sum of the digits of a two-digit number is 7. If 27 is added to the number the result is expressed by the digits in reverse order. What was the original number?

28. A man's age is now five sevenths of what it will be 10 years from now. What is his present age?

29. A field containing 10,000 square feet has a length twice its width. What are the dimensions of the field?

30. The altitude of a triangle is two thirds the base. If the area is 24 square feet, what are the base and altitude?

CHAPTER XXII

QUADRATIC EQUATIONS

129. Introduction. The simplest quadratic equation is one in which the term of the first degree is lacking; as, for example, $x^2 - 4 = 0$ or $x^2 + 41 = 0$, where the terms in x are lacking. In order to solve these equations one term should be transposed (if necessary) so as to bring one term on each side of the equation. The square root of each member should be taken, not neglecting the $\pm$ sign before the square root of the constant term. Equations like the foregoing are sometimes called *pure quadratics*.

ORAL EXERCISES

Solve the following pure quadratic equations :

1. $x^2 - 9 = 0$. 7. $y^2 = 1$. 13. $x^2 - 16 a^2 = 0$.

2. $x^2 - 25 = 0$. 8. $y^2 - 64 = 0$. 14. $x^2 - 9 c^2 = 0$.

3. $y^2 - 16 = 0$. 9. $x^2 - 4 = 12$. 15. $y^2 - 25 b^4 = 0$.

4. $x^2 - 1 = 0$. 10. $-y^2 + 16 = 0$. 16. $25 = m^2$.

5. $y^2 - 4 = 0$. 11. $x^2 - 4 a^2 = 0$. 17. $4 a^2 - 9 x^2 = 0$.

6. $-x^2 + 4 = 0$. 12. $y^2 - 25 b^4 = 0$. 18. $-169 + x^2 = 0$.

Before taking up the work that follows, the student should review the method of forming trinomial squares given on page 152.

ORAL EXERCISES

What terms should be added in order to make the following expressions perfect trinomial squares?

- | | | |
|--------------------|-----------------------------|------------------------------|
| 1. $x^2 + 2x + ?$ | 6. $x^2 - 8x + ?$ | 11. $y^2 - 9y + ?$ |
| 2. $y^2 + 6y + ?$ | 7. $x^2 + 3x + ?$ | 12. $x^2 + \frac{2}{3}x + ?$ |
| 3. $x^2 - 4x + ?$ | 8. $r^2 + r + ?$ | 13. $x^2 + \frac{2}{5}x + ?$ |
| 4. $y^2 - 2y + ?$ | 9. $x^2 - \frac{1}{2}x + ?$ | 14. $x^2 + \frac{3}{4}x + ?$ |
| 5. $x^2 + 16x + ?$ | 10. $x^2 + 5x + ?$ | 15. $t^2 - \frac{3}{2}t + ?$ |

130. Solution by completing the square. The method of solving quadratic equations by completing the square depends on the rearrangement of the terms of the equation so that the left-hand member is a trinomial and a perfect square and the right-hand member is a constant.

In order that the left-hand member may become such a perfect square, it is usually necessary to add a properly chosen number to each member of the equation.

EXAMPLES

1. Solve $x^2 + 2x - 15 = 0$. (1)

Solution. Transposing, $x^2 + 2x = 15$. (2)

Adding 1 to each member of (2),

$$x^2 + 2x + 1 = 16. \quad (3)$$

Then

$$(x + 1)^2 = 4^2. \quad (4)$$

Extracting the square root of each member of (4),

$$x + 1 = \pm 4. \quad (5)$$

Whence $x = -1 + 4 = 3$

and $x = -1 - 4 = -5$.

Check. Substituting 3 for x in (1), $9 + 6 - 15 = 0$, or $0 = 0$.

Substituting -5 for x in (1), $25 - 10 - 15 = 0$, or $0 = 0$.

$$2. \text{ Solve } 2x^2 + 3x - 2 = 0. \quad (1)$$

$$\text{Solution. Transposing, } 2x^2 + 3x = 2. \quad (2)$$

Dividing (2) by the coefficient of x^2 ,

$$x^2 + \frac{3}{2}x = 1. \quad (3)$$

Adding $(\frac{3}{4})^2$ to each member of the equation,

$$x^2 + \frac{3}{2}x + (\frac{3}{4})^2 = \frac{9}{16} + \frac{16}{16}. \quad (4)$$

$$\text{Then} \quad (x + \frac{3}{4})^2 = (\frac{5}{4})^2. \quad (5)$$

$$\text{Extracting the square root, } x + \frac{3}{4} = \pm \frac{5}{4}. \quad (6)$$

$$x = -\frac{3}{4} \pm \frac{5}{4}$$

and

$$x = \frac{1}{2} \text{ or } -2.$$

Check. Substituting $\frac{1}{2}$ for x in (1),

$$2(\frac{1}{4}) + 3(\frac{1}{2}) - 2 = 0$$

$$\frac{1}{2} + \frac{3}{2} - 2 = 0, \text{ or } 0 = 0.$$

Substituting -2 for x in (1),

$$2(-2)^2 + 3(-2) - 2 = 0.$$

$$8 - 6 - 2 = 0, \text{ or } 0 = 0.$$

The method of solving a quadratic equation in x illustrated in the preceding examples may be stated in the

RULE. Transpose terms so that the terms containing x are in the first member and those which do not contain x are in the second.

Divide both members of the equation by the coefficient of x^2 unless that coefficient is $+1$.

Then add to both members the square of half the coefficient of x (in the equation just obtained), thus making the first member a perfect trinomial square.

Rewrite the equation, expressing the first member as the square of a binomial and the second member in its simplest form.

Extract the square root of both members of the equation and write the sign $\pm$ before the square root of the second member, thus obtaining two linear equations.

Solve the equation in which the second member is taken with the sign + and then solve the equation in which the second member is taken with the sign —. The results are the roots of the quadratic.

CHECK. *Substitute each result separately in place of x in the original equation. If the resulting equations are not obvious identities, simplify until they become so.*

Before attempting to solve quadratic equations by the method of completing the square, it is often desirable to observe whether they can be solved by factoring according to the method set forth on page 172, since it is frequently simpler to solve a quadratic equation by factoring than to solve it by the method of completing the square.

EXERCISES

Solve by completing the square, and check as directed by the teacher :

1. $x^2 + 6x - 7 = 0.$

14. $5x^2 + 13x - 6 = 0.$

2. $x^2 + 4x = 12.$

15. $6x^2 - 5x + 1 = 0.$

3. $y^2 - 10y = 24.$

16. $5x^2 - x - 4 = 0.$

4. $a^2 + 8a - 105 = 0.$

17. $x^2 = 15.$

5. $m^2 - 2m = 80.$

18. $6x^2 - 5x - 6 = 0.$

6. $x^2 - 6x = 16.$

19. $4x^2 + 13x + 9 = 0.$

7. $2x^2 - 7x + 5 = 0.$

20. $5y^2 + 22y = 15.$

8. $x(x + 4) - 3(x + 4) = 0.$

21. $21x^2 - 2x - 8 = 0.$

9. $2x^2 - 3x + 1 = 0.$

22. $20x^2 - 43x + 14 = 0.$

10. $2x^2 + x - 10 = 0.$

23. $12m^2 - 67m = 50.$

11. $3x^2 + 15x + 10 = 3x - 2.$

24. $30y^2 - y - 1 = 0.$

12. $3y^2 + 7y = -4.$

25. $135x^2 - 42x = 16.$

13. $y^2 - 13 = 0.$

26. $42x^2 - 51x + 15 = 0.$

27. Why is not equation (5), Example 1, page 326, written with the sign $\pm$ before each member?

$$28. 3x^2 - 7x + 1 = 0. \quad (1)$$

$$\text{Solution. Transposing, } 3x^2 - 7x = -1, \quad (2)$$

Dividing each member of (2) by 3,

$$x^2 - \frac{7}{3}x = -\frac{1}{3}. \quad (3)$$

Adding $(-\frac{7}{6})^2$ to each member of (3),

$$x^2 - \frac{7}{3}x + \frac{49}{36} = \frac{49}{36} - \frac{12}{36} = \frac{37}{36}. \quad (4)$$

Then

$$(x - \frac{7}{6})^2 = \frac{37}{36}. \quad (5)$$

$$x - \frac{7}{6} = \pm \frac{1}{6}\sqrt{37}. \quad (6)$$

$$x = \frac{7}{6} \pm \frac{1}{6}\sqrt{37} \quad (7)$$

$$= \frac{7}{6} \pm \frac{6.0828}{6} \quad (8)$$

$$= \frac{13.0828}{6} \text{ or } \frac{.9172}{6} \quad (9)$$

$$= 2.180 \text{ or } 0.153. \quad (10)$$

Check. Since the values in (10) are not *exact* values of x , they will, if substituted for x in (1), make its first member *nearly* but not exactly equal to zero.

An exact check on the radical forms of the roots can be obtained by substituting from equation (7) in equation (1). This check may be shortened by substituting both roots at the same time, as follows:

Substituting $\frac{7}{6} \pm \frac{1}{6}\sqrt{37}$ for x in $3x^2 - 7x + 1 = 0$,

$$\text{we get } 3(\frac{7}{6} \pm \frac{1}{6}\sqrt{37})^2 - 7(\frac{7}{6} \pm \frac{1}{6}\sqrt{37}) + 1 = 0,$$

$$3(\frac{49}{36} \pm \frac{14}{36}\sqrt{37} + \frac{37}{36}) - 7(\frac{7}{6} \pm \frac{1}{6}\sqrt{37}) + 1 = 0,$$

$$\frac{49}{12} \pm \frac{14}{12}\sqrt{37} + \frac{37}{12} - \frac{49}{6} \mp \frac{7}{6}\sqrt{37} + 1 = 0.$$

The radical terms vanish because the two upper signs before them must be taken together, and then the two lower signs.

$$\text{Therefore,} \quad \frac{4}{12} + \frac{3}{12} - \frac{9}{12} + \frac{1}{12} = 0,$$

$$\text{or} \quad \frac{9}{12} - \frac{9}{12} = 0.$$

In quadratic equations like the preceding the radical forms of the roots are often sufficient; at other times values to two or three decimal places are necessary. Unless otherwise directed, obtain only the radical forms of irrational roots.

In the following exercises obtain correct to three decimal places the values of any radical answers which may occur:

$$29. x^2 - 5x + 2 = 0.$$

$$30. \frac{3}{4} - x^2 = 3x.$$

$$31. 3x + 5x^2 = 14.$$

$$32. 4x + 15 = 25x^2.$$

$$33. 6t^2 - 5t = 21.$$

$$34. 5x^2 - 20x = -19.$$

$$35. 2x^2 - 15 = x^2 - 2x - 7.$$

$$36. x^2 = 21x - 5.$$

$$37. 14x^2 + 5x = 2.$$

$$38. 3z^2 + \frac{2}{3}z - 2 = 0.$$

$$39. x^2 + 3.4x + 1.7 = 0.$$

$$40. x^2 - 2.1x - .6 = 0.$$

$$41. .2x^2 + .7x - .42 = 0.$$

$$42. .5x^2 - 2.3x + .4 = 0.$$

$$43. x^2 - \frac{4\sqrt{3}}{3}x = 5.$$

$$44. \frac{a}{3} + 4 - \frac{15}{a} = 0.$$

$$45. \frac{x}{5} - \frac{2}{3x} - 1 = 0.$$

$$46. \frac{2}{t-2} - \frac{3t}{2} = 0.$$

$$47. x = \frac{3}{x+2}.$$

$$48. \frac{4}{y+5} = -3y.$$

$$49. \frac{1+x}{2+x} - \frac{1}{6} = \frac{3-x}{5-x}.$$

$$50. \frac{1}{2x+4} = \frac{15}{4} + \frac{1}{x-2}.$$

$$51. \frac{y+2}{y-4} + \frac{y-1}{y+5} + 1 = 0.$$

BIOGRAPHICAL NOTE

KARL FRIEDRICH GAUSS. Standing in the very front rank of mathematicians, with Archimedes and Newton, is Karl Friedrich Gauss (1777-1855). He was the son of a bricklayer and was afforded an education, much against the will of his parents, by a nobleman who had noticed his remarkable talents.

At the age of nineteen he had made discoveries which had baffled all mathematicians up to that time, and continued to make important contributions to the theories of mathematics and astronomy for the next fifty years.

The student has undoubtedly noticed that quadratic equations have two roots. It is not difficult to prove that cubic equations have three roots and that equations of the fourth degree have four roots. That any equation in one unknown has a number of roots equal to its degree was first proved by Gauss. He gave three distinct proofs of this fact, although no one before his time had been able to prove it at all.

In collaboration with another professor at the University of Göttingen, he invented the telegraph independently of the American, S. F. B. Morse, and probably earlier.

REVIEW EXERCISES

Solve, and check as directed by the teacher :

1. $10x^2 - 11x + 1 = 0$.

3. $x + 2 = 25x^2$.

2. $11x^2 - 5x = 7$.

4. $9x^2 = 15x + 2$.

5. $(4x - 3)(x + 2) = (3x + 25)(2x + 3)$.

6. $(3x + 4)(x - 1) = (5x + 2)(3x - 2)$.

7. $5x^2 + 11x - 12 = 0$.

10. $\frac{91x}{17} - 3 = \frac{22}{17x}$.

8. $\frac{4x}{3} = 5 + \frac{25}{x}$.

11. $\frac{b}{5} = \frac{10}{b} + 12$.

9. $\frac{21}{x} = x + 2$.

12. $\frac{x^2}{x + 2} = \frac{5}{4}$.

$$13. \frac{x}{3} = \frac{9+2x}{x}. \quad 14. \frac{2x}{x+1} + \frac{10x}{15} = 25. \quad 15. \frac{3x^2}{x+1} = \frac{5x}{3}.$$

$$16. (14x-5)(3x+2) = (21x+5)(3x-2).$$

$$17. \frac{x+1}{x-5} + \frac{x+2}{x-3} = 5. \quad 21. 2x^2 - 5x + 2 = 0.$$

$$18. \frac{5x-3}{5x+2} = \frac{1}{2} - \frac{2x-1}{3x+4}. \quad 22. \frac{13x^2}{5x+1} = \frac{2x}{13} + 2.$$

$$19. \frac{x}{5-x} = \frac{3}{4} + \frac{3}{x}. \quad 23. \frac{2x+1}{5x+3} + \frac{1}{12} - \frac{x+1}{x-1} = 0.$$

$$20. y = \frac{21}{2y+3}. \quad 24. \frac{x+2}{5x-1} - \frac{1}{15} = \frac{x+2}{x+1}.$$

PROBLEMS

(Reject all answers which do not satisfy the conditions of the problems.)

1. The square of a certain number added to three times the same number is equal to 40. What is the number?

2. One half the square of a number is equal to six more than twice the number. Find the number.

3. What are the two consecutive numbers whose product is 812?

4. What are the two consecutive odd numbers whose product is 483?

5. What are the two consecutive even numbers whose product is 2208?

6. What are the three consecutive numbers such that the product of the last two is equal to the sum of all three?

7. The area of a rectangular field is 1 square mile. The length is 400 feet greater than the width. Find the dimensions.

8. Two square fields have a total area of 10,000 square feet. The side of one field is 50 feet longer than the side of the other. What are the dimensions of the two fields?

9. The diagonal of a rectangle is 50 feet. If one side is 20 feet longer than the other, what are the dimensions?

10. One side of a right triangle is two thirds of the other. The hypotenuse is 28 feet. Find the length of the two sides.

11. The same number expresses both the perimeter of a square in feet and its area in square feet. Find the side of the figure.

12. An automobile made a trip of 125 miles out and 125 miles back in 11 hours. The average speed on the return trip was 3 miles an hour less than on the outward trip. What was the rate in each direction?

13. Two trains run 384 miles. The faster train has a speed 4 miles per hour greater than the speed of the other and requires 1 hour less time. What is the speed of each train?

14. The edges of two cubical boxes differ by 2 inches. The volumes differ by 56 cubic inches. Find the dimensions of the boxes.

15. Two pumps can fill a tank in 45 minutes. One pump takes 10 minutes longer to fill it alone than the other one does. What is the time that each pump alone requires to fill the tank?

16. If the price of eggs is raised 25 cents per dozen, \$3 will buy 24 fewer eggs than would have been obtained at the original price. What was the original price?

17. A rectangular garden is made up of a flower-bed 10 feet longer than it is wide, surrounded by a walk 3 feet wide. The area of the bed is 204 square feet less than the area of the total garden. Find the dimensions of the bed and of the whole garden.

The velocity of a body falling from rest varies as the expression $V = 32t$, where V is the velocity in feet per second and t is the time in seconds that the body has been falling. The distance traveled by the same body is expressed by the equation $S = 16t^2$, where S is the distance traveled in feet and t , as before, is the time in seconds.

18. How long will it take for a stone dropped from an elevation of 2100 feet to reach the ground?

19. What will be the velocity of a body which has fallen a distance of 1000 feet, neglecting the resistance of the air?

20. A man drops a stone down a well and hears the splash 2.59 seconds later. If the sound is heard .09 seconds after the splash is seen, how deep is the well? How fast does the sound travel in feet per second?

History of the quadratic equation. Though the development of the method of solving quadratic equations is closely connected with the general growth of algebra, yet it is possible to indicate rather briefly the most important steps in the process.

The first writer on formal algebra was Diophantos, who lived at Alexandria, in Egypt, about A. D. 275. Most of his work that is preserved is devoted to the solution of problems that lead to equations. So far as we know he was the first to indicate the unknown number by a single letter, in this respect being far in advance of many mathematicians who lived much later. It is a little remarkable, in fact, that so able and original a man as Diophantos should have exerted so little influence on his successors. He solved his quadratic equations by a method not unlike that of completing the square, but his imperfect knowledge of the nature of numbers made it impossible

for him to understand the entire significance of the process. Though he made every effort not to consider equations whose roots were not positive integers, sometimes they would creep in, and under such circumstances, when his method led him to a negative or irrational root, he rejected the whole equation as absurd or impossible. Even when both the roots were positive he took only the one afforded by the positive sign in the formula for solving a quadratic.

The difficulties of Diophantos are typical of those encountered by mathematicians for the next fifteen hundred years. The difficulty lay not in finding a formal method of solving the equation but in understanding the result after it was obtained. The meanings of negative and of imaginary numbers (that is, even roots of negative numbers) have been two of the most difficult of all mathematical ideas for men to grasp.

Five or six hundred years later the Hindus devised a general solution of the quadratic, but their chief advance over Diophantos lay in the fact that they did not regard an equation whose roots were negative as necessarily absurd, but merely rejected the negative result with the remark, "It is inadequate; people do not approve of negative roots." The Hindus, however, did realize that a quadratic equation sometimes has two roots, a fact that Diophantos never comprehended.

No material gain in the understanding of the solutions of the quadratic can be found until the seventeenth century. The keenest mathematicians of the sixteenth century, like Cardan and Vieta, rejected negative roots, though by this time irrational roots were admitted. In fact, in 1544 Stifel, a German, published an algebra in which irrational numbers are included among the numbers proper. But he affirms that except in the case where a quadratic equation has two positive roots no equation has more than one root. It was not until the work of Descartes and Gauss became widely known that the nature of the roots of all kinds of quadratic equations was completely understood.

131. Systems involving a linear and a quadratic equation.

A *quadratic equation* in two unknowns contains one or more terms of the second degree, but no term of higher degree in those unknowns. Every system of equations in two unknowns in which one equation is *linear* and the other *quadratic* can be solved by the method of *substitution*.

EXAMPLE

$$\text{Solve the system } \begin{cases} 2y^2 + x^2 = 6, & (1) \\ 3x - y = 5. & (2) \end{cases}$$

Solution. Solving (2) for y in terms of x ,

$$y = 3x - 5. \quad (3)$$

Substituting $3x - 5$ for y in (1),

$$2(3x - 5)^2 + x^2 = 6. \quad (4)$$

$$\text{From (4),} \quad 19x^2 - 60x + 44 = 0. \quad (5)$$

$$\text{Solving (5),} \quad x = 2 \text{ or } \frac{2}{19}. \quad (6)$$

Substituting 2 for x in (3),

$$y = 1.$$

Substituting $\frac{2}{19}$ for x in (3),

$$y = -\frac{2}{19}.$$

The two sets of roots are therefore $x = 2, y = 1$, and $x = \frac{2}{19}, y = -\frac{2}{19}$.

Check. Substituting 2 for x and 1 for y in (1) and (2),

$$2 + 4 = 6,$$

$$6 - 1 = 5.$$

Substituting $\frac{2}{19}$ for x and $-\frac{2}{19}$ for y in (1) and (2),

$$\frac{1}{3} \frac{6}{6} \frac{2}{1} + \frac{4}{3} \frac{8}{6} \frac{4}{1} = \frac{2}{3} \frac{1}{6} \frac{6}{1} = 6,$$

$$\frac{6}{1} \frac{6}{9} + \frac{2}{1} \frac{9}{9} = \frac{9}{1} \frac{5}{9} = 5.$$

The similarity between this method and that for the solution of linear systems by substitution should be carefully noted.

The method given in the preceding solution for solving a system consisting of one quadratic and one linear equation is stated in the

RULE. *Solve the linear equation for one unknown in terms of the other.*

Substitute this value in the quadratic equation and solve the resulting equation.

Substitute each of the roots of the quadratic equation thus found in the linear equation, and solve, thus obtaining two sets of roots of the simultaneous system.

CHECK. *Substitute as usual, in both equations.*

EXERCISES

Solve the following systems, pair results, and check each set of roots:

$$\begin{array}{lll} 1. \quad 2x^2 + y = 33, & 4. \quad x + y^2 = 19, & 7. \quad 4m - n^2 = -28, \\ & 3x + y = 13. & n - 7m = -8. \end{array}$$

$$\begin{array}{lll} 2. \quad x^2 + y^2 = 13, & 5. \quad x^2 - 5y = 4, & 8. \quad 5x^2 + y^2 = 30, \\ 3x + 2y = 12. & x - 5y = -38. & x + 2y = 11. \end{array}$$

$$\begin{array}{lll} 3. \quad x^2 - y = 1, & 6. \quad 2x^2 + 5y^2 = 125, & 9. \quad 6A^2 + 7B^2 = 82, \\ y - x = 1. & x + 3y = 15. & 5A - 4B = 7. \end{array}$$

$$\begin{array}{ll} 10. \quad x^2 + 5xy = 14, & 12. \quad x^2 + 3xy - 2y^2 = 188, \\ x + 2y = 4. & 2x = 5y. \end{array}$$

$$\begin{array}{ll} 11. \quad x^2 + 5xy = 76, & 13. \quad 5x^2 + 2xy + 3y^2 = 195, \\ x - y = 1. & 2x + y = 13. \end{array}$$

PROBLEMS

(Reject all results which do not satisfy the conditions of the problems.)

1. The sum of two numbers is 5. The sum of their squares is 125. Find the numbers.

2. The difference of two numbers is 25. The sum of their squares is 925. What are the numbers?

3. A rectangular playing field is 180 feet longer than it is wide. Its area is 49,500 square feet. Find the dimensions.

4. The area of one square field is nine sixteenths that of another. The perimeter of one is 20 feet less than that of the other. What are the dimensions?

5. A park is a right triangle. One of the two shorter sides is 25 feet longer than the other. The area is 625 square feet. What are the dimensions?

6. The annual income from a certain investment is \$200. If the principal were \$500 less and the interest rate 1% more, the income would be \$25 more. What was the amount of the principal and what was the rate of interest?

7. The perimeter of a rectangle is 26 inches, and the area is 42 square inches. Find the dimensions.

8. The diagonal of a rectangle is 25 inches and its perimeter is 70 inches. What are its dimensions?

9. The area of a rectangle is 150 square feet. The perimeter of the figure is 50 feet. Determine the dimensions.

10. The speed of a boat is twice the speed of the current in a river. The boat heads straight across the stream and at the end of 5 minutes is .93 miles from its starting point. What is the speed of the boat and of the current?

11. The perimeter of a rectangle and the area are both expressed by the number 18. What are the dimensions?

12. A motorist leaves a point A and travels north. At the same time a second motorist, who travels 50% faster than the first, leaves a point 3 miles east of A and travels east. Half an hour later the distance between them is 17 miles. Find the rate of travel of each motorist.

REVIEW EXERCISES

Factor the following expressions :

1. $6x^2 + 13xy + 6y^2$.

2. $8x^2 + 12x + 2ax + 3a$.

3. $2mny + 6mnx + 9mx^2 + 3mxy + 9nx^2 + 3nxy$
 $+ 6n^2x + 2n^2y$.

Solve for the unknowns involved and check :

4. $x^2 - 5 = 0$.

8. $3x + 5y = -7$,
 $2x + 3y - 25 = -29$.

5. $a^2 + 3a = 5$.

$\frac{2x}{3} + \frac{5y}{2} = -8\frac{2}{3}$,

6. $3m^2 - 2m - 15 = 0$.

9. $\frac{3x}{4} - \frac{2y}{5} = 3\frac{1}{10}$.

7. $3x + 2y - 5 = 14$,
 $25x^2 + 2xy + 3y^2 = 657$.

10. $.5x + .23y = .61$,
 $.33y - .54x = 3.39$.

Divide :

11. $10x^4 + 31x^3y + 26x^2y^2 - xy^3 - 6y^4$ by $x + y$.

12. $14x^2y^2 - 3xy^3 + 2y^4 + 15$ by $x - 3y$.

Find the square root of the following expressions :

13. 25,403.25.

14. 37,281.05.

15. 200.453.

16. $4 - 12a + 9a^2 + 2(2x - 3ax - 6ay + 4y) + x^2$
 $+ 4xy + 4y^2$.

17. $4x^2 + 12xy + 16x + 9y^2 + 24y + 16$.

18. A workman has a sawhorse with legs each 30 inches long, arranged like an inverted V. The horse stands 24 inches high. The man finds that he can adjust the height of the horse by varying the distance between its legs. How much must he decrease the distance between the legs to raise the horse 2 inches?

19. A man standing with his feet together is 6 feet tall. When he stands with his feet 2 feet apart his apparent height is 5.83 feet. How long are his legs?

20. The perimeter of a rectangle is 14 feet and its diagonal is 5 feet. What is its area?

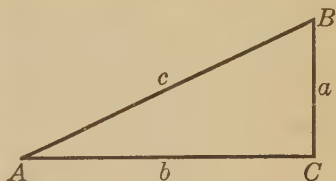
21. A collection of dimes and quarters is worth \$6.55. There are one more than eight times as many quarters as dimes. How many coins of each kind are there?

22. Two trains start from points 100 miles apart and travel toward each other until they meet. One train has a speed of 40 miles per hour and the other has a speed of 35 miles per hour. What are the distances between the meeting point and the points from which they originally started?

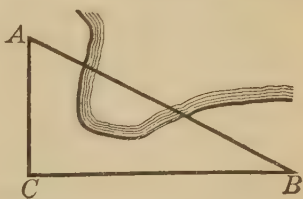
SUPPLEMENTARY CHAPTER

INTRODUCTION TO TRIGONOMETRY

132. Introduction. The right triangle ABC has three angles and three sides, which constitute the six *parts* of the triangle. In the sections that follow, methods will be explained for finding the remaining parts of any right triangle if the length of one side and the size of another part (other than the right angle) is known. That this problem is of great practical importance can be seen from the following illustration:



If A and B are two points near the shore of a lake but separated by a bay, the distance between them cannot be measured directly. But if a point C can be found such that the angle ACB is a right angle, and if the line AC and the angle A can be measured, the methods that follow will enable us to find the length of AB . It is a simple matter for a surveyor to find the point C . Hence the distance AB can be ascertained.



133. Facts from geometry. Before going further it is necessary to state a few facts about the right triangle, some of which have already been used in arithmetic.

1. A right angle contains 90 degrees.
2. The sum of the angles of any triangle is 180 degrees.
3. The sum of the two acute angles of a right triangle is 90 degrees.

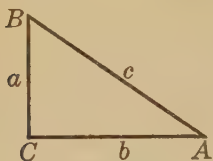


FIG. 1

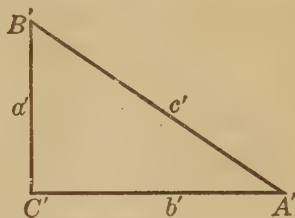


FIG. 2

4. In a right triangle (Fig. 1), $c^2 = a^2 + b^2$ when a , b , and c denote the lengths of the sides and c is opposite the right angle.

5. If two triangles, ABC and $A'B'C'$ (Figs. 1 and 2), have their like-lettered angles equal, then the ratios of like-lettered sides are equal. That is, $\frac{a}{a'} = \frac{c}{c'}$, $\frac{b}{b'} = \frac{c}{c'}$, etc.

The queries 1-4 below refer to Fig. 1:

QUERY 1. If the angle $A = 70^\circ$, how many degrees are there in angle B ? If angle $A = 27^\circ$? If angle $A = 12^\circ$?

QUERY 2. Can any angle of a right triangle contain more than 90° ?

QUERY 3. If $a = 12$ inches and $b = 5$ inches, how long is c ?

QUERY 4. If $b = 1$ foot and $c = 15$ inches, how long is a ?

QUERY 5. It is necessary to know many distances that cannot be measured directly but must be determined somewhat as in the illustration of § 132. Give some examples of such distances.

134. Trigonometric ratios. In order to attack the problem stated on page 341 it is necessary to denote the ratios of the sides of a right triangle by certain names.

Let ABC be any right triangle with the right angle at C .

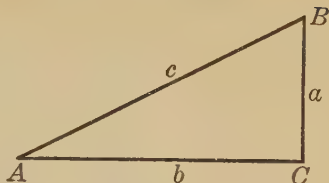
The *sine* of the angle $A = \frac{a}{c} = \frac{\text{side opposite}}{\text{hypotenuse}}$.

The *cosine* of the angle A

$$= \frac{b}{c} = \frac{\text{side adjacent}}{\text{hypotenuse}}.$$

The *tangent* of the angle A

$$= \frac{a}{b} = \frac{\text{side opposite}}{\text{side adjacent}}.$$



These definitions are abbreviated as follows:

$$\sin A = \frac{a}{c}; \quad \cos A = \frac{b}{c}; \quad \tan A = \frac{a}{b}.$$

It is exceedingly important to note that the letters a , b , and c denote the *lengths* of the sides, all in terms of the same unit of measure. Thus, if $a = 6$ inches and $c = 2$ feet, $\sin A = \frac{6}{24} = .25$.

QUERY 1. What is $\sin A$, $\cos A$, $\tan A$, if $a = 3$, $b = 4$, $c = 5$?

QUERY 2. Can $\sin A$ be greater than 1? If so, why? If not, why not?

QUERY 3. Can $\cos A$ be greater than 1? If so, why? If not, why not?

QUERY 4. Can $\tan A$ be greater than 1? If so, why? If not, why not?

QUERY 5. Give two different sets of values of a and b if $\tan A = 2$; if $\tan A = 5$.

QUERY 6. Give two different sets of values of a and c if $\sin A = \frac{2}{3}$; if $\sin A = \frac{4}{5}$.

135. Tables of trigonometric ratios. The values of the trigonometric ratios for all angles have been computed to many decimal places. The table on page 344 gives the values of the sine, the cosine, and the tangent correct to three places.

QUERY 1. From an inspection of the table should you say that $\sin A$ increases or decreases in value as angle A becomes larger?

THREE-PLACE TABLE OF NATURAL FUNCTIONS

Angle	Sin	Cos	Tan	Angle	Sin	Cos	Tan
1°	.017	.9998	.017	45°	.707	.707	1.000
2°	.035	.9994	.035	46°	.719	.695	1.036
3°	.052	.9986	.052	47°	.731	.682	1.072
4°	.070	.9976	.070	48°	.743	.669	1.111
5°	.087	.996	.087	49°	.755	.656	1.150
6°	.105	.995	.105	50°	.766	.643	1.192
7°	.122	.993	.123	51°	.777	.629	1.235
8°	.139	.990	.141	52°	.788	.616	1.280
9°	.156	.988	.158	53°	.799	.602	1.327
10°	.174	.985	.176	54°	.809	.588	1.376
11°	.191	.982	.194	55°	.819	.574	1.428
12°	.208	.978	.213	56°	.829	.559	1.483
13°	.225	.974	.231	57°	.839	.545	1.540
14°	.242	.970	.249	58°	.848	.530	1.600
15°	.259	.966	.268	59°	.857	.515	1.664
16°	.276	.961	.287	60°	.866	.500	1.732
17°	.292	.956	.306	61°	.875	.485	1.804
18°	.309	.951	.325	62°	.883	.469	1.881
19°	.326	.946	.344	63°	.891	.454	1.963
20°	.342	.940	.364	64°	.899	.438	2.050
21°	.358	.934	.384	65°	.906	.423	2.144
22°	.375	.927	.404	66°	.914	.407	2.246
23°	.391	.921	.424	67°	.921	.391	2.356
24°	.407	.914	.445	68°	.927	.375	2.475
25°	.423	.906	.466	69°	.934	.358	2.605
26°	.438	.899	.488	70°	.940	.342	2.747
27°	.454	.891	.510	71°	.946	.326	2.904
28°	.469	.883	.532	72°	.951	.309	3.078
29°	.485	.875	.554	73°	.956	.292	3.271
30°	.500	.866	.577	74°	.961	.276	3.487
31°	.515	.857	.601	75°	.966	.259	3.732
32°	.530	.848	.625	76°	.970	.242	4.011
33°	.545	.839	.649	77°	.974	.225	4.331
34°	.559	.829	.675	78°	.978	.208	4.705
35°	.574	.819	.700	79°	.982	.191	5.145
36°	.588	.809	.727	80°	.985	.174	5.671
37°	.602	.799	.754	81°	.988	.156	6.314
38°	.616	.788	.781	82°	.990	.139	7.115
39°	.629	.777	.810	83°	.993	.122	8.144
40°	.643	.766	.839	84°	.995	.105	9.514
41°	.656	.755	.869	85°	.996	.087	11.43
42°	.669	.743	.900	86°	.9976	.070	14.30
43°	.682	.731	.933	87°	.9986	.052	19.08
44°	.695	.719	.966	88°	.9994	.035	28.64
45°	.707	.707	1.000	89°	.9998	.017	57.29

QUERY 2. Should you say that $\cos A$ increases or decreases as A becomes larger?

QUERY 3. Verify from the table $\sin 32^\circ = .530$; $\cos 32^\circ = .848$; $\tan 32^\circ = .625$; $\sin 67^\circ = .921$; $\tan 56^\circ = 1.483$; $\cos 84^\circ = .105$.

QUERY 4. Read from the table

$\sin 4^\circ$; $\sin 28^\circ$; $\sin 47^\circ$; $\sin 76^\circ$.
 $\cos 89^\circ$; $\cos 35^\circ$; $\cos 60^\circ$; $\cos 8^\circ$.
 $\tan 45^\circ$; $\tan 71^\circ$; $\tan 5^\circ$; $\tan 86^\circ$.

QUERY 5. From the table determine the number of degrees in angle A if

(a) $\sin A = .438$	(d) $\cos A = .966$	(g) $\tan A = 1.732$
(b) $\sin A = .982$	(e) $\cos A = .225$	(h) $\tan A = .445$
(c) $\sin A = .707$	(f) $\cos A = .755$	(i) $\tan A = 4.011$

QUERY 6. From the table determine the number of degrees in the angle that most nearly satisfies the following:

(a) $\sin A = .841$	(d) $\cos A = .983$	(g) $\tan A = .262$
(b) $\sin A = .969$	(e) $\cos A = .425$	(h) $\tan A = .799$
(c) $\sin A = .180$	(f) $\cos A = .815$	(i) $\tan A = 3.200$

QUERY 7.

$\sin 30^\circ = ?$	$\cos 60^\circ = ?$
$\sin 47^\circ = ?$	$\cos 43^\circ = ?$
$\sin 20^\circ = ?$	$\cos (90^\circ - 20^\circ) = ?$
$\sin 38^\circ = ?$	$\cos (90^\circ - 38^\circ) = ?$

QUERY 8. If $A + B = 90^\circ$, what relation exists between $\sin A$ and $\cos B$? $\cos A$ and $\sin B$?

QUERY 9. From $\sin A = \frac{a}{c}$ express a in terms of $\sin A$ and c .

QUERY 10. From $\cos A = \frac{b}{c}$ find b ; find c .

QUERY 11. Solve $\tan A = \frac{a}{b}$ for a ; for b .

QUERY 12. Solve $\cos 27^\circ = \frac{b}{90}$ for b .

HINT. From the table, $\cos 27^\circ = .891$.

QUERY 13. Solve $\tan 40^\circ = \frac{12}{b}$ for b .

QUERY 14. Solve $\sin 63^\circ = \frac{42}{c}$ for c .

186. Use of trigonometric ratios. If one side and one acute angle of a right triangle can in any way be determined, the other parts of the triangle may be computed by means of the trigonometric ratios.

EXAMPLES

1. In the right triangle ABC the angle $A = 24^\circ$ and $b = 42$. Find BC .

Solution. $\tan A = \frac{a}{b}$.

$$\tan 24^\circ = \frac{a}{42}$$

$$a = 42 \times \tan 24^\circ \\ = 42(.445) = 18.69.$$

Then a , or BC , = 18.69.



From the definitions of the trigonometric ratios on pages 342-343 it appears that each definition contains three elements, two sides and one angle. Hence, if any two of these are known, the third can be found.

For example, if a and c are known, $\sin A$ can be found, and from the table the value of the angle A can be determined approximately.

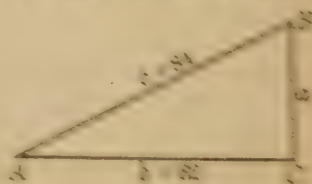
Since the sum of the acute angles of a right triangle is 90° , if one is known the other can be found immediately.

2. In the right triangle ABC , $b = 32$, $c = 54$. Find A .

Solution. $\cos A = \frac{b}{c}$.

$$\cos A = \frac{32}{54} = .593.$$

$A = 42^\circ$ approximately.



EXERCISES

(Exercises 1-9, inclusive, refer to right triangles.)

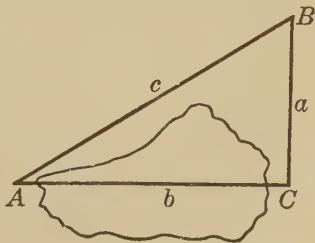
1. Given $A = 25^\circ$, $c = 28$. Find B , a , and b .
2. Given $A = 69^\circ$, $b = 57$. Find B , a , and c .
3. Given $A = 47^\circ$, $a = 21$. Find B , b , and c .
4. Given $B = 73^\circ$, $c = 51$. Find A , a , and b .
5. Given $B = 31^\circ$, $b = 100$. Find A , a , and c .
6. Given $B = 8^\circ$, $a = 84$. Find A , b , and c .
7. Given $A = 27^\circ$, $b = 48$. Find a , B , and c .
8. Given $A = 84^\circ$, $c = 90$. Find a , B , and b .
9. Given $B = 65^\circ$, $c = 147$. Find A , b , and a .

10. The shadow of a tree on level ground is 60 feet long, and the sun's rays make an angle of 52° with the ground. How high is the tree?

11. A ladder 50 feet long rests against the side of a house and reaches a point on the house 30 feet from the ground. What angle does it make with the ground?

12. If the edge of the Great Pyramid is 609 feet long and it makes an angle of 52° with the horizontal, what is the height of the pyramid?

13. A surveyor finds that the angle B in the adjacent figure is 54° , $C = 90^\circ$, and the line AB is found to be 627 feet long. How long is the line AC ?



14. A ship sails southwest a distance of 36 miles. How far is it south of the starting point?

15. A flagstaff 85 feet high casts a shadow 123 feet long. What is the angle of elevation of the sun?

16. A ladder rests against the side of a building, making an angle of 30° with the ground. The foot of the ladder is 18 feet from the building. How long is the ladder? How high is the top of the ladder?

137. Four-place tables of trigonometric ratios. For many purposes more accurate numeric results are desired than it is possible to obtain by the use of three-place tables. If the measurement of lines is made with an accuracy that insures only three correct places, no more accurate tables are needed, for numeric results cannot be attained with any greater degree of accuracy than the least accurate of the measurements with which one starts. It is possible, however, both in the laboratory and in the field to take measurements so accurately that four-place tables are desirable.

Such tables are found on pages 352-354. These tables give the three ratios already defined for every ten minutes of angle.

By the process of interpolation it is possible to find the values of these ratios exact to the nearest minute.

Interpolation depends on the assumption that the change in the ratio is proportional to the change in the corresponding angle for the 10-minute intervals.

There are two problems of interpolation :

(a) Given an angle not in the table, to find the corresponding sine, cosine, or tangent.

(b) Given the value of a sine, a cosine, or a tangent not in the table, to find the corresponding angle.

EXAMPLES

1. What are the values of $\sin 34^\circ 25'$ and of $\sin 34^\circ 28'$?

Solution. From the table we find that $\sin 34^\circ 20' = .5640$ and $\sin 34^\circ 30' = .5664$. We assume that $\sin 34^\circ 25'$ lies just halfway between .5640 and .5664, or at .5652. $\sin 34^\circ 28'$ would be .8 of the way from .5640 to .5664, or at .5659.

A similar procedure should be followed in finding the tangent or the cosine of an angle correct to minutes.

When the numeric value of the ratio is given and it is desired to find the angle to the nearest minute, the following operation is employed.

2. It is desired to find A when $\sin A = .4423$.

Solution. Reference to the tables shows that since .4423 lies between .4410 and .4436, A must lie between $26^\circ 10'$ and $26^\circ 20'$. The entire difference between .4410 and .4436, which is 26 in the last two places, corresponds to the entire $10'$ between $26^\circ 10'$ and $26^\circ 20'$. But .4423 is 13 in the last two places greater than .4410. Hence this 13 corresponds to $\frac{13}{26}$, or $\frac{1}{2}$, of the whole $10'$ between $26^\circ 10'$ and $26^\circ 20'$. That is, $A = 26^\circ 15'$.

Similar work may be abbreviated as follows:

Given $\tan A = 1.2443$. Find A .

$$\tan 51^\circ 20' = 1.2497; \tan A = 1.2443$$

$$\tan 51^\circ 10' = 1.2423; \tan 51^\circ 10' = 1.2423$$

74

20

$$\frac{20}{74} \text{ of } 10' = 3'.$$

Hence

$$A = 51^\circ 13'.$$

In interpolating with the cosine ratio it is necessary to remember that as the angle increases the cosine decreases. Hence the correction should be subtracted from the value of the cosine of the smaller angle instead of being added to it.

EXERCISES

Find from the tables on pages 352-354 :

- | | | |
|--------------------------|--------------------------|--------------------------|
| 1. $\sin 18^\circ 40'$. | 3. $\tan 40^\circ 10'$. | 5. $\tan 70^\circ 50'$. |
| 2. $\cos 36^\circ 20'$. | 4. $\cos 60^\circ 30'$. | 6. $\sin 80^\circ 20'$. |

Verify the following statements :

- | | |
|-----------------------------------|------------------------------------|
| 7. $\sin 35^\circ 25' = .5795$. | 11. $\cos 63^\circ 17' = .4496$. |
| 8. $\sin 53^\circ 5' = .7995$. | 12. $\cos 41^\circ 33' = .7484$. |
| 9. $\tan 25^\circ 42' = .4813$. | 13. $\tan 61^\circ 34' = 1.8469$. |
| 10. $\tan 41^\circ 38' = .8889$. | 14. $\cos 15^\circ 2' = .9658$. |

138. The cotangent. It is often a convenience in practical work to use a fourth ratio, called the *cotangent*. With reference to the figure of § 134,

$$\cot A = \frac{b}{a} = \frac{\text{side adjacent}}{\text{side opposite}}.$$

139. Explanation of the arrangement of the tables. From the definitions of § 134 it is clear that $\sin 31^\circ$ equals $\cos 59^\circ$ and $\sin 62^\circ$ equals $\cos 28^\circ$. Further, if two sides of a right triangle are equal, the angles opposite are equal and each is 45° . Hence $\sin 45^\circ$ equals $\cos 45^\circ$. It follows from these simple relations that the sine of any angle between 0° and 45° equals the cosine of an angle between 45° and 90° , and the sine of an angle between 45° and 90° equals the cosine of an angle between 0° and 45° . Advantage is taken of this relation in the arrangement of the tables, and the number of pages necessary is half what it would otherwise be.

Referring to the tables on pages 352-354, the sines of angles less than 45° are obtained by noting the given angle in the left-hand column, and the required ratio is the number opposite the angle in the column headed SIN. Similarly for the cosine, tangent, and cotangent.

If an angle is between 45° and 90° , the angle is read from the right-hand column and the sine is the number opposite in the column at the *foot* of which is the word SIN. Similarly for the cosine.

For the tangent and the cotangent a similar method holds. This follows from their definition. A brief inspection of the tables will verify this statement.

EXERCISES

Find by interpolation :

- | | | |
|--------------------------|--------------------------|--------------------------|
| 1. $\sin 12^\circ 18'$. | 4. $\sin 75^\circ 12'$. | 7. $\cos 18^\circ 18'$. |
| 2. $\tan 35^\circ 42'$. | 5. $\cos 18^\circ 15'$. | 8. $\cot 24^\circ 48'$. |
| 3. $\tan 80^\circ 36'$. | 6. $\cos 18^\circ 12'$. | 9. $\cot 72^\circ 56'$. |

In Exercises 10-18 the triangles are understood to be right triangles :

10. Given $A = 40^\circ 15'$, $c = 100$. Find a .
11. Given $B = 32^\circ 24'$, $a = 60$. Find b .
12. Given $A = 64^\circ 18'$, $c = 28$. Find a .
13. Given $B = 54^\circ 32'$, $b = 40$. Find c .
14. Given $a = 32$, $c = 51$. Find A .
15. Given $b = 45$, $c = 80$. Find B .
16. Given $a = 110$, $b = 4$. Find A .
17. Given $a = 3$, $b = 4$. Find the angles and hypotenuse.
18. Given $c = 26$, $a = 10$. Find the other parts.

°	SIN	COS	TAN	COT		°	SIN	COS	TAN	COT	
0°	.0000	1.0000	.0000	∞	90°	8°	.1392	.9903	.1405	7.1154	82°
10'	.0029	1.0000	.0029	343.77	50'	10'	.1421	.9899	.1435	6.9682	50'
20'	.0058	1.0000	.0058	171.89	40'	20'	.1449	.9894	.1465	6.8269	40'
30'	.0087	1.0000	.0087	114.59	30'	30'	.1478	.9890	.1495	6.6912	30'
40'	.0116	.9999	.0116	85.940	20'	40'	.1507	.9886	.1524	6.5606	20'
50'	.0145	.9999	.0145	68.750	10'	50'	.1536	.9881	.1554	6.4348	10'
1°	.0175	.9998	.0175	57.290	89°	9°	.1564	.9877	.1584	6.3138	81°
10'	.0204	.9998	.0204	49.104	50'	10'	.1593	.9872	.1614	6.1970	50'
20'	.0233	.9997	.0233	42.964	40'	20'	.1622	.9868	.1644	6.0844	40'
30'	.0262	.9997	.0262	38.188	30'	30'	.1650	.9863	.1673	5.9758	30'
40'	.0291	.9996	.0291	34.368	20'	40'	.1679	.9858	.1703	5.8708	20'
50'	.0320	.9995	.0320	31.242	10'	50'	.1708	.9853	.1733	5.7694	10'
2°	.0349	.9994	.0349	28.636	88°	10°	.1736	.9848	.1763	5.6713	80°
10'	.0378	.9993	.0378	26.432	50'	10'	.1765	.9843	.1793	5.5764	50'
20'	.0407	.9992	.0407	24.542	40'	20'	.1794	.9838	.1823	5.4845	40'
30'	.0436	.9990	.0437	22.904	30'	30'	.1822	.9833	.1853	5.3955	30'
40'	.0465	.9989	.0466	21.470	20'	40'	.1851	.9827	.1883	5.3093	20'
50'	.0494	.9988	.0495	20.206	10'	50'	.1880	.9822	.1914	5.2257	10'
3°	.0523	.9986	.0524	19.081	87°	11°	.1908	.9816	.1944	5.1446	79°
10'	.0552	.9985	.0553	18.075	50'	10'	.1937	.9811	.1974	5.0658	50'
20'	.0581	.9983	.0582	17.169	40'	20'	.1965	.9805	.2004	4.9894	40'
30'	.0610	.9981	.0612	16.350	30'	30'	.1994	.9799	.2035	4.9152	30'
40'	.0640	.9980	.0641	15.605	20'	40'	.2022	.9793	.2065	4.8430	20'
50'	.0669	.9978	.0670	14.924	10'	50'	.2051	.9787	.2095	4.7729	10'
4°	.0698	.9976	.0699	14.301	86°	12°	.2079	.9781	.2126	4.7046	78°
10'	.0727	.9974	.0729	13.727	50'	10'	.2108	.9775	.2156	4.6382	50'
20'	.0756	.9971	.0758	13.197	40'	20'	.2136	.9769	.2186	4.5736	40'
30'	.0785	.9969	.0787	12.706	30'	30'	.2164	.9763	.2217	4.5107	30'
40'	.0814	.9967	.0816	12.251	20'	40'	.2193	.9757	.2247	4.4494	20'
50'	.0843	.9964	.0846	11.826	10'	50'	.2221	.9750	.2278	4.3897	10'
5°	.0872	.9962	.0875	11.430	85°	13°	.2250	.9744	.2309	4.3315	77°
10'	.0901	.9959	.0904	11.059	50'	10'	.2278	.9737	.2339	4.2747	50'
20'	.0929	.9957	.0934	10.712	40'	20'	.2306	.9730	.2370	4.2193	40'
30'	.0958	.9954	.0963	10.385	30'	30'	.2334	.9724	.2401	4.1653	30'
40'	.0987	.9951	.0992	10.078	20'	40'	.2363	.9717	.2432	4.1126	20'
50'	.1016	.9948	.1022	9.7882	10'	50'	.2391	.9710	.2462	4.0611	10'
6°	.1045	.9945	.1051	9.5144	84°	14°	.2419	.9703	.2493	4.0108	76°
10'	.1074	.9942	.1080	9.2553	50'	10'	.2447	.9696	.2524	3.9617	50'
20'	.1103	.9939	.1110	9.0098	40'	20'	.2476	.9689	.2555	3.9136	40'
30'	.1132	.9936	.1139	8.7769	30'	30'	.2504	.9681	.2586	3.8667	30'
40'	.1161	.9932	.1169	8.5555	20'	40'	.2532	.9674	.2617	3.8208	20'
50'	.1190	.9929	.1198	8.3450	10'	50'	.2560	.9667	.2648	3.7760	10'
7°	.1219	.9925	.1228	8.1443	83°	15°	.2588	.9659	.2679	3.7321	75°
10'	.1248	.9922	.1257	7.9530	50'	10'	.2616	.9652	.2711	3.6891	50'
20'	.1276	.9918	.1287	7.7704	40'	20'	.2644	.9644	.2742	3.6470	40'
30'	.1305	.9914	.1317	7.5958	30'	30'	.2672	.9636	.2773	3.6059	30'
40'	.1334	.9911	.1346	7.4287	20'	40'	.2700	.9628	.2805	3.5656	20'
50'	.1363	.9907	.1376	7.2687	10'	50'	.2728	.9621	.2836	3.5261	10'
	COS	SIN	COT	TAN	°		COS	SIN	COT	TAN	°

°	SIN	COS	TAN	COT		°	SIN	COS	TAN	COT	
16°	.2756	.9613	.2867	3.4874	74°	24°	.4067	.9135	.4452	2.2460	66°
10'	.2784	.9605	.2899	3.4495	50'	10'	.4094	.9124	.4487	2.2286	50'
20'	.2812	.9596	.2931	3.4124	40'	20'	.4120	.9112	.4522	2.2113	40'
30'	.2840	.9588	.2962	3.3759	30'	30'	.4147	.9100	.4557	2.1943	30'
40'	.2868	.9580	.2994	3.3402	20'	40'	.4173	.9088	.4592	2.1775	20'
50'	.2896	.9572	.3026	3.3052	10'	50'	.4100	.9075	.4628	2.1609	10'
17°	.2924	.9563	.3057	3.2709	73°	25°	.4226	.9063	.4663	2.1445	65°
10'	.2952	.9555	.3089	3.2371	50'	10'	.4253	.9051	.4699	2.1283	50'
20'	.2979	.9546	.3121	3.2041	40'	20'	.4279	.9038	.4734	2.1123	40'
30'	.3007	.9537	.3153	3.1716	30'	30'	.4305	.9026	.4770	2.0965	30'
40'	.3035	.9528	.3185	3.1397	20'	40'	.4331	.9013	.4806	2.0809	20'
50'	.3062	.9520	.3217	3.1084	10'	50'	.4358	.9001	.4841	2.0655	10'
18°	.3090	.9511	.3249	3.0777	72°	26°	.4384	.8988	.4877	2.0503	64°
10'	.3118	.9502	.3281	3.0475	50'	10'	.4410	.8975	.4913	2.0353	50'
20'	.3145	.9492	.3314	3.0178	40'	20'	.4436	.8962	.4950	2.0204	40'
30'	.3173	.9483	.3346	2.9887	30'	30'	.4462	.8949	.4986	2.0057	30'
40'	.3201	.9474	.3378	2.9600	20'	40'	.4488	.8936	.5022	1.9912	20'
50'	.3228	.9465	.3411	2.9319	10'	50'	.4514	.8923	.5059	1.9768	10'
19°	.3256	.9455	.3443	2.9042	71°	27°	.4540	.8910	.5095	1.9626	63°
10'	.3283	.9446	.3476	2.8770	50'	10'	.4566	.8897	.5132	1.9486	50'
20'	.3311	.9436	.3508	2.8502	40'	20'	.4592	.8884	.5169	1.9347	40'
30'	.3338	.9426	.3541	2.8239	30'	30'	.4617	.8870	.5206	1.9210	30'
40'	.3365	.9417	.3574	2.7980	20'	40'	.4643	.8857	.5243	1.9074	20'
50'	.3393	.9407	.3607	2.7725	10'	50'	.4669	.8843	.5280	1.8940	10'
20°	.3420	.9397	.3640	2.7475	70°	28°	.4695	.8829	.5317	1.8807	62°
10'	.3448	.9387	.3673	2.7228	50'	10'	.4720	.8816	.5354	1.8676	50'
20'	.3475	.9377	.3706	2.6985	40'	20'	.4746	.8802	.5392	1.8546	40'
30'	.3502	.9367	.3739	2.6746	30'	30'	.4772	.8788	.5430	1.8418	30'
40'	.3529	.9356	.3772	2.6511	20'	40'	.4797	.8774	.5467	1.8291	20'
50'	.3557	.9346	.3805	2.6279	10'	50'	.4823	.8760	.5505	1.8165	10'
21°	.3584	.9336	.3839	2.6051	69°	29°	.4848	.8746	.5543	1.8040	61°
10'	.3611	.9325	.3872	2.5826	50'	10'	.4874	.8732	.5581	1.7917	50'
20'	.3638	.9315	.3906	2.5605	40'	20'	.4899	.8718	.5619	1.7796	40'
30'	.3665	.9304	.3939	2.5386	30'	30'	.4924	.8704	.5658	1.7675	30'
40'	.3692	.9293	.3973	2.5172	20'	40'	.4950	.8689	.5696	1.7556	20'
50'	.3719	.9283	.4006	2.4960	10'	50'	.4975	.8675	.5735	1.7437	10'
22°	.3746	.9272	.4040	2.4751	68°	30°	.5000	.8660	.5774	1.7321	60°
10'	.3773	.9261	.4074	2.4545	50'	10'	.5025	.8646	.5812	1.7205	50'
20'	.3800	.9250	.4108	2.4342	40'	20'	.5050	.8631	.5851	1.7090	40'
30'	.3827	.9239	.4142	2.4142	30'	30'	.5075	.8616	.5890	1.6977	30'
40'	.3854	.9228	.4176	2.3945	20'	40'	.5100	.8601	.5930	1.6864	20'
50'	.3881	.9216	.4210	2.3750	10'	50'	.5125	.8587	.5969	1.6753	10'
23°	.3907	.9205	.4245	2.3559	67°	31°	.5150	.8572	.6009	1.6643	59°
10'	.3934	.9194	.4279	2.3369	50'	10'	.5175	.8557	.6048	1.6534	50'
20'	.3961	.9182	.4314	2.3183	40'	20'	.5200	.8542	.6088	1.6426	40'
30'	.3987	.9171	.4348	2.2998	30'	30'	.5225	.8526	.6128	1.6319	30'
40'	.4014	.9159	.4383	2.2817	20'	40'	.5250	.8511	.6168	1.6212	20'
50'	.4041	.9147	.4417	2.2637	10'	50'	.5275	.8496	.6208	1.6107	10'
	COS	SIN	COT	TAN	°		COS	SIN	COT	TAN	°

°	SIN	COS	TAN	COT		°	SIN	COS	TAN	COT	
32°	.5299	.8480	.6249	1.6003	58°	40°	.6428	.7660	.8391	1.1918	50°
10'	.5324	.8465	.6289	1.5900	50'	10'	.6450	.7642	.8441	1.1847	50'
20'	.5348	.8450	.6330	1.5798	40'	20'	.6472	.7623	.8491	1.1778	40'
30'	.5373	.8434	.6371	1.5697	30'	30'	.6494	.7604	.8541	1.1708	30'
40'	.5398	.8418	.6412	1.5597	20'	40'	.6517	.7585	.8591	1.1640	20'
50'	.5422	.8403	.6453	1.5497	10'	50'	.6539	.7566	.8642	1.1571	10'
33°	.5446	.8387	.6494	1.5399	57°	41°	.6561	.7547	.8693	1.1504	49°
10'	.5471	.8371	.6536	1.5301	50'	10'	.6583	.7528	.8744	1.1436	50'
20'	.5495	.8355	.6577	1.5204	40'	20'	.6604	.7509	.8796	1.1369	40'
30'	.5519	.8339	.6619	1.5108	30'	30'	.6626	.7490	.8847	1.1303	30'
40'	.5544	.8323	.6661	1.5013	20'	40'	.6648	.7470	.8899	1.1237	20'
50'	.5568	.8307	.6703	1.4919	10'	50'	.6670	.7451	.8952	1.1171	10'
34°	.5592	.8290	.6745	1.4826	56°	42°	.6691	.7431	.9004	1.1106	48°
10'	.5616	.8274	.6787	1.4733	50'	10'	.6713	.7412	.9057	1.1041	50'
20'	.5640	.8258	.6830	1.4641	40'	20'	.6734	.7392	.9110	1.0977	40'
30'	.5664	.8241	.6873	1.4550	30'	30'	.6756	.7373	.9163	1.0913	30'
40'	.5688	.8225	.6916	1.4460	20'	40'	.6777	.7353	.9217	1.0850	20'
50'	.5712	.8208	.6959	1.4370	10'	50'	.6799	.7333	.9271	1.0786	10'
35°	.5736	.8192	.7002	1.4281	55°	43°	.6820	.7314	.9325	1.0724	47°
10'	.5760	.8175	.7046	1.4193	50'	10'	.6841	.7294	.9380	1.0661	50'
20'	.5783	.8158	.7089	1.4106	40'	20'	.6862	.7274	.9435	1.0599	40'
30'	.5807	.8141	.7133	1.4019	30'	30'	.6884	.7254	.9490	1.0538	30'
40'	.5831	.8124	.7177	1.3934	20'	40'	.6905	.7234	.9545	1.0477	20'
50'	.5854	.8107	.7221	1.3848	10'	50'	.6926	.7214	.9601	1.0416	10'
36°	.5878	.8090	.7265	1.3764	54°	44°	.6947	.7193	.9657	1.0355	46°
10'	.5901	.8073	.7310	1.3680	50'	10'	.6967	.7173	.9713	1.0295	50'
20'	.5925	.8056	.7355	1.3597	40'	20'	.6988	.7153	.9770	1.0235	40'
30'	.5948	.8039	.7400	1.3514	30'	30'	.7009	.7133	.9827	1.0176	30'
40'	.5972	.8021	.7445	1.3432	20'	40'	.7030	.7112	.9884	1.0117	20'
50'	.5995	.8004	.7490	1.3351	10'	50'	.7050	.7092	.9942	1.0058	10'
37°	.6018	.7986	.7536	1.3270	53°	45°	.7071	.7071	1.0000	1.0000	45°
10'	.6041	.7969	.7581	1.3190	50'						
20'	.6065	.7951	.7627	1.3111	40'						
30'	.6088	.7934	.7673	1.3032	30'						
40'	.6111	.7916	.7720	1.2954	20'						
50'	.6134	.7898	.7766	1.2876	10'						
38°	.6157	.7880	.7813	1.2799	52°						
10'	.6180	.7862	.7860	1.2723	50'						
20'	.6202	.7844	.7907	1.2647	40'						
30'	.6225	.7826	.7954	1.2572	30'						
40'	.6248	.7808	.8002	1.2497	20'						
50'	.6271	.7790	.8050	1.2423	10'						
39°	.6293	.7771	.8098	1.2349	51°						
10'	.6316	.7753	.8146	1.2276	50'						
20'	.6338	.7735	.8195	1.2203	40'						
30'	.6361	.7716	.8243	1.2131	30'						
40'	.6383	.7698	.8292	1.2059	20'						
50'	.6406	.7679	.8342	1.1988	10'						
	COS	SIN	COT	TAN	°		COS	SIN	COT	TAN	°

Solve the following right triangles :

- | | |
|--|---------------------------------------|
| 19. $A = 30^\circ$, $c = 1325$. | 24. $b = 1888$, $A = 41^\circ 37'$. |
| 20. $B = 45^\circ$, $a = 856$. | 25. $b = 318$, $a = 250$. |
| 21. $c = 63.7$, $A = 27^\circ 15'$. | 26. $c = 423$, $b = 184$. |
| 22. $c = 3.428$, $B = 50^\circ 44'$. | 27. $c = 1296$, $a = 1024$. |
| 23. $a = .0725$, $A = 72^\circ 52'$. | 28. $b = 124.5$, $A = 23.6^\circ$. |

29. The Eiffel Tower is about 986 feet high. At a certain time it casts a shadow 1200 feet long. What angle do the sun's rays then make with the ground?

30. A boat travels southwest 318.4 miles. It is then how many miles south of its starting point?

31. A balloon is anchored by a rope 875 feet long. When the air is quiet a man in the balloon observes that the line from his eye to an object on the ground makes an angle of $30^\circ 20'$ with the horizontal. How far is he from the object?

32. Two men on a level field observed a balloon and found its direction with respect to the horizontal to be 40° and 62° respectively. The balloon was 4000 feet high and in the same vertical plane with the two men. How far apart were the men?

33. The sides a and b of a triangle (not a right triangle) are 200 feet and 156 feet respectively. Their included angle C is 50° . A line is drawn from A , making a right angle with the opposite side. This line is an altitude of the triangle. Find it. Find the area of the triangle.

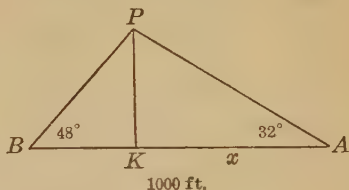
34. Two sides of a triangle are 120 feet and 144 feet respectively, and their included angle is 75° . Find one altitude and the area of the triangle.

35. From two points A and B on shore 1000 feet apart a boat P is observed. The angle $BAP = 32^\circ$ and the angle $ABP = 48^\circ$. Find the length of the perpendicular from P to the line AB .

HINT. Draw the triangle ABP and the perpendicular PK . Let $AK = x$. Then $BK = 1000 - x$, and

$\tan 32^\circ = \frac{PK}{x}$ (1), and $\tan 48^\circ =$

$\frac{PK}{1000 - x}$ (2). Solve (1) and (2).



36. Find AP and BP in Exercise 35.

37. Assuming that the Mt. Washington railway rises 3596 feet in traveling along 3 miles of track, what average angle does the track make with the horizontal?

38. A tree 75 feet high stands on the bank of a river and subtends an angle of $32^\circ 42'$ on the bank directly opposite. What is the width of the river?

39. A guy-rope from the top of a vertical 120-foot pole is attached to the ground 45 feet from the base of the pole. What angle does the rope make with the ground, assuming the ground to be horizontal?

40. A trolley car climbs 50 feet in traveling along a half mile of track. What is the average angle between the track and the horizontal?

41. An airplane is above a point 2 miles from an observer. The line of sight from the observer to the airplane makes an angle of $25^\circ 42'$ with the horizontal. How high is the airplane above the ground?

42. A railway embankment is 8 feet high. Its width at the top is 10 feet and at the bottom is 25 feet. What angle do the sides make with the horizontal?

43. A tree falls against a building 15 feet distant. If the top of the tree rests against the building at a point 25 feet from the ground, what angle does the trunk of the tree make with the horizontal? How tall was the tree?

44. From the level of the roof of a building 100 feet high a man observes the top of a taller building which is 200 feet distant from him. The line of sight from the observer to the top of the second building makes an angle of $36^{\circ} 52'$ with the horizontal. What is the height of the second building? How far is the observer from the roof of the taller building?

45. It is desired to run a wire from a point on a house 15 feet from the ground to the top of a 25-foot pole, which is at a distance of 50 feet from the house. How much wire will be needed assuming that it can be stretched in a straight line? What angle will the wire make with the horizontal? What amount must be added to the height of the pole so as to double the angle between the wire and the horizontal? If the wire is allowed to sag will more or less wire be required to go between the house and the pole?

46. In certain states the highway commissions will not allow public roads to be built with a gradient steeper than 1 foot of rise for every 10 feet of roadway. What angle does such a road make with the horizontal?

47. A motorbus has a route in the form of a loop. The bus starts from the highest point in the loop and descends 350 feet in half its route. It then returns by another road to its starting point, traveling in all 2 miles. What is the angle between the roadway and the horizontal for each half of the trip? for the route as a whole?

SUPPLEMENTARY EXERCISES AND PROBLEMS

CHAPTER I

INTRODUCTION

Exercises

1. Find K in $K = 2 \pi R H$, if $\pi = \frac{22}{7}$, $R = 20$, $H = 14$.
2. Find A in $A = \frac{ab}{2}$, if $a = 10$ and $b = 18$.
3. Find A in $A = 2 \pi r h$, if $\pi = \frac{22}{7}$, $r = 4$, $h = 14$.
4. Find C in $C = 2 \pi r$, if $\pi = \frac{22}{7}$, $r = 21$.
5. Find d in $d = rt$, if $r = 40$, $t = 18$.
6. Find a in $a = \frac{AD}{360}$, if $A = 80$ and $D = 72$.
7. Find C in $C = \frac{360}{D}$, if $D = 8$.
8. Find I in $I = \frac{E}{R + r}$, if $E = 84$, $R = 17$, $r = 11$.
9. Find I in $I = \frac{E}{R + nr}$, if $E = 252$, $R = 18$, $n = 2$, $r = 9$.
10. Find I in $I = \frac{ne}{R + nr}$, if $n = 9$, $e = 7$, $R = 13$, $r = 4$.
11. Find C in $C = \frac{5}{9}(F - 32)$, if $F = 95$.
12. Find F in $F = \frac{9}{5}C + 32$, if $C = 20$.
13. Find W_1 in $W_1 = \frac{L_2 W_2}{L_1}$, if $L_2 = 12$, $W_2 = 80$, $L_1 = 7.5$.
14. Find A in $A = P(1 + rt)$, if $P = 500$, $r = .05$, $t = 2$.
15. Find A in $A = P(1 + r)^t$, if $P = 100$, $r = .04$, $t = 2$.
16. Find V_1 in $V_1 = \frac{V_2 P_2}{P_1}$, if $V_2 = 140$, $P_2 = 24$, $P_1 = 15$.

17. Find S in $S = \frac{n(a+l)}{2}$, if $n = 28$, $a = 3$, $l = 59$.
 18. Find S in $S = \frac{rl-a}{r-1}$, if $r = 2$, $l = 57$, $a = 5$.
 19. Find a in $a = \frac{Dn}{180}$, if $D = 70$, $n = 30$.
 20. Find V_1 in $V_1 = V_0(1 + .00365 t_1)$, if $V_0 = 50$, $t_1 = 10$.
 21. Find A in $A = \frac{(b_1 + b_2)a}{2}$, if $b_1 = 18$, $b_2 = 24$, $a = 12.5$.
 22. Find f in $\frac{1}{f} = \frac{1}{a} + \frac{1}{b}$, if $a = 8$, $b = 12$.
 23. Find A in $A = \pi r^2$, if $\pi = 3.1416$, $r = 5$.
 24. Find V in $V = lwh$, if $l = 20$, $w = 12$, $h = 6$.
 25. Find V in $V = \frac{\pi r^2 h}{3}$, if $r = 10$, $h = 15$, $\pi = \frac{22}{7}$.
 26. Find V in $V = \frac{4}{3} \pi r^3$, if $r = 4$, $\pi = \frac{22}{7}$.
 27. Find I in $I = Prt$, if $P = 250$, $r = .06$, $t = 1.5$.
 28. Find s in $s = \frac{1}{2} gt^2$, if $g = 32$ and $t = 9$.
- If $a = 8$, $b = 3$, $c = 12$, $d = \frac{1}{2}$, find the value of :
29. $a + \frac{2ab}{cd} \cdot \frac{c}{b} \div a + 5dc \div b - 3ad$.
 30. $(abcd - ab) \div c + 2bd - 5\frac{ab}{c} + 15$.

CHAPTER III

POSITIVE AND NEGATIVE NUMBERS

Exercises

1. Find C in $C = \frac{5}{9}(F - 32)$, if $F = -13$.
2. Find F in $F = \frac{9}{5}C + 32$, if $C = -20$.
3. Find S in $S = \frac{rl-a}{r-1}$, if $r = -2$, $l = -45$, $a = 3$.

4. Find l in $l = a + (n - 1)d$, if $a = 5$, $n = 27$, $d = -3$.
5. Find S in $S = \frac{n}{2}(a + l)$, if $n = 11$, $a = 2$, $l = -56$.
6. Find S in $S = \frac{n}{2}[2a + (n - 1)d]$, if $n = 8$, $a = 7$, $d = -2$.

If x is 3, find value of :

7. $(2x - 1) - 3(x - 1)$. 8. $(5 - 2x) - (5 - x)$.
9. $x(1 - 2x) + (7 - 3x)$.

If x is -4 , find the value of :

10. $2x(1 - 3x) + (2 - x)(x - 4)$.
11. $(7 - 2x) - (3x + 1)x$.
12. $(3x - 1) - (2x - 1)$.
13. Does $x^2 - 3x + 2 = 12$, if $x = -2$? if $x = -4$?
14. Does $2x^2 - 7x + 1 = -2$, if $x = -1$? if $x = 3$?
15. Does $2x^2 - x = -3$, if $x = -5$? if $x = 3$?

If x is -2 and y is 5, find the value of :

16. $x^2 - 2xy + y^2$. 17. $2x^2 + y - 2xy^2$.
18. $2y^2 + 4x - x^2$.

CHAPTER IV

ADDITION

Exercises

Simplify by collecting similar terms :

1. $3a - 2b + 5c - a - 2c + 10b$.
2. $7b - 6c - a - 4a - 3b + c$.
3. $-2x + y - z - 11x - 17z$.
4. $5ac - 6bc + ab - 13bc - 16ab + 9ac$.
5. $8xy - 23yz - 17xz + 22xz + 6xy + 7yz$.
6. $3ax - 9by + 14ax - 7cz + 21by$.

7. $m - 2n + 6am - 42m - 7n$.
8. $7am - 6bn + 3ax + 5bn - 4am - 5ax$.
9. $x^2 - 2x - 4 - 6x + 17 - 5x^2$.
10. $2a^2x + 3ax + 9ax^2 - 6ax - 14a^2x - ax^2$.
11. $4a^2b - 8x^2y - 2bc^2 + 3a^2b + x^2y$.
12. $2r^3s - 11rs^3 - rs - r^3s + 9rs^3 + rs$.
13. $2(r + s) + 7(a + b) - 5(a + b) - 3(r + s) + 2(r + s)$.
14. $6a(x + y) - 5(b - c) + 2a(x + y) + 2(b - c)$.
15. $2x^3 + 4x^2 - 3(x + y) - 6x^2 - 5(x + y) + 9x^3$.

Find the sum of the following polynomials and check results:

16. $a - 2b + c$, $-5a + 8c - 2b$, and $7b - 6a + 9c$.
17. $-4r + 6s - 2t$, and $7s + 9t - 2r$.
18. $ar - 11bs - 5ct$, and $-4bs - 7ct - 6ar$.
19. $4ab - 6bc$, $8ac - 9abc$, $9bc - 5ab$, $2abc + ac$, and $2bc - 4abc$.
20. $2a^2 - ab + b^2$, $2ab - a^2 + a$, and $4b^2 - 2ab - a^2$.
21. $7u^2 - 2uv$, $8v^2 - 2uv^2$, $-6uv + 2v^2$, and $4uv^2 - 5u^2$.
22. $2ax^2 - 3by^2 + cz^2$, $-7ax^2 - 2by^2 - 6cz^2$, and $3ax^2 + by^2 + 2cz^2$.
23. $p^2 - 8pq - q^2$, $4pq - 2p^2 + q^2$, and $7pq - 5p^2 + 4q^2$.
24. $2mn^3 - 4mn - m^3n$, and $-5mn + 2m^3n - 7mn^3$.
25. $2x^2y - 5yz^2 - xy^2$, $8xy^2 - 5x^2y + 2yz^2$, and $2yz^2 - 2xy^2 + x^2y$.

CHAPTER V

SIMPLE EQUATIONS

Exercises

Solve the following equations and check:

1. $2x - 5 = 15$.
2. $7x + 2 = 23$.
3. $4y = 21 + y$.
4. $4t - 2 = 18 - t$.
5. $6z - 13 = 2 - z$.
6. $2w + 8 = w + 11$.

7. $12 + 3x = 2 + x.$

8. $11 + 7y = y - 4.$

9. $3 - 2k = k + 15.$

10. $m + 6 = 3m - 10.$

11. $\frac{2r}{3} = 5 - r.$

12. $\frac{x}{2} - 11 = 4.$

13. $8 + \frac{3s}{5} = s.$

14. $3d + 12d + 17 - d + 4d = 125.$

15. $4h - 4 + 3h = 21 + 2h + 5.$

16. $7 - 2z + 3 = z - 9 - 4z.$

17. $5 + 2(3 - 4x) = 17 - x + 8.$

18. $k + 10 = 4 - 3(2 + k).$

Problems

Review
 ① The first of three numbers is twice the second, and the third is 5 more than three times the second. Their sum is 65. Find the numbers. *10-35-20*

2. The first of two numbers is four times the second. Their difference is 51. What are the numbers?

3. The perimeter of a parallelogram is 70 feet. Its length is 2 feet more than twice its width. Find its dimensions.

4. Seven times a certain number exceeds twice itself by 70. What is the number?

5. The length of one piece of rope is three times that of a second piece. If 7 feet be taken from the longer and added to the shorter, the two pieces will be of equal length. Find the length of the pieces.

6. A farmer wishes to grind some grain. In order to have three times as much shelled corn as rye and 5 more sacks of oats than rye, how many sacks of each must he take to the mill so that he will have 65 sacks of feed after it is ground?

7. A flagpole 86 feet high was broken by the wind. The part left standing was 2 feet shorter than one third the part broken off. How long was each part?

CHAPTER VI

SUBTRACTION

Exercises

Subtract the first polynomial from the second and check:

1. $a - 3b - 2c$, $2a - 4b + 3c$.
2. $2x + y - 3z$, $2y - x + z$.
3. $6x^2 - 4x + 8$, $2x^2 + 3x - 7$.
4. $3a^2 - 2ab - b^2$, $2b^2 + ab - a^2$.
5. $5x^2 + 6xy - 2$, $4x^2 - 11$.
6. $m^3 + 2mn - 6$, $4mn - 2m^2 + 4$.
7. $2g^3 - 5gh^2 - 4$, $g^3 + g^2h + 2$.
8. $3x^2 - 2xy - by^2$, $ax^2 - by^2 - 3$.
9. $2r^2s^2 - r^3$, $r^3 - 3r^2s^2 + s^3$.
10. $2v + 6x - 2r$, $3s - 4x - 5$.

In Exercises 11-18 find the expression which, added to the first, will give the second:

11. $5h - 11k - 4$, $3h - 7k - 8$.
12. $2ax^2 + by^2 - cz^2$, $6ax^2 - 2by^2 + c^2$.
13. $p^3 - 3p^2q + pq^2$, $4p^2q - pq^2 - 2q^3$.
14. $2ab^2 - 3a^4 + a^2b^4$, $a^4 - 7ab^2 - 6a^2b^2$.
15. $2x^3 + 7x - 9$, $6x^2 - x^3 + 5$.
16. $m^2n^2 - 3m^3 + 8n^2$, $m^3 + 7m^2n^2 - 2n^3$.
17. $4ak - 7bl - cm$, $9bl - 8cm + dn$.
18. $4r^2s^2 + 5r^3 + st^2$, $7r^3 - t^3$.
19. From the sum of $2x - y + 7z$ and $2y + z - x$, take $5x + z - 2y$.
20. From the sum of $3a - 2b - d$ and $7c - 8d + 4a$, take $3a - 6c + 4d - 5b$.
21. From the sum of $2x - 3v + 9$ and $4a + x - 5$, take $11 - 3v + x$.

22. From the sum of $3ax - 4by + cz$ and $2ax - 7cz$, take $6 + ax - 6by$.

23. From $2g - 3h + 9k$ take the sum of $4h + 7k - 2$ and $3k + 5h + 2l$.

24. From $x^3 - 7x^2y + 2xy^2$ take the sum of $4x^2y - 2xy^2 + 4y^3$ and $5x^2y + 2x^3 + 3xy^2$.

25. From the sum of $5a^3 + 2a^2bc + 6b^2c$ and $-3a^2bc + 4b^2c + b^3$, take $4a^2bc - 7b^2c - a^3 + 2b^3$.

CHAPTER VII.

IDENTITIES AND EQUATIONS

Exercises

Solve and check:

1. $5x - 3 = 2x + 9$. 4. $5r + 12 + 3r = 9 + 2r$.

2. $4y + 2 = 14 - 2y$. 5. $7 - 2a = 15 + 4a - 2$.

3. $10 - 6x = 3x + 9$. 6. $13 + 2n - 3 = 7n - 6 + n$.

7. $2h - 11 - 5h + 9 = 7$.

8. $5 - 2k - 17 + 3k = 15 - 4k + 8$.

9. $9 - 4x + 13 - 7x = 2x - 21 - 7x$.

10. $6 - 2m - 23 + 5m = 19 + 6m - 4 + 13m$.

11. $3p - 17 - 8p + 23 - 6p = -22$.

12. $4t + 15 - 10t + 37 = 7t - 4$

13. $5 - 9q + 28 + 3q - 13 + 12q + 23 = 0$.

14. $14s + 6 = -8s + 11 + 5s + 35 + 2s$.

15. $-8 + 4b - 15 - 21b - 39 + 8b + 11 = 0$.

16. $0 = -9 + 8y + 26 - 20y + 14 + 7y$.

17. $3v^2 - 18 + 10v = 3v^2 + 19v + 9$.

18. $15 - 2a^2 - 14a + 22 = 7 - 6a - 2a^2$.

19. $0 = 22 - 14x + 31 + 25x - 23 - 6x$.

20. $14 - 3x - 2x^2 + 8 = 4x - 15 - 2x^2$.

Problems

1. Four times a certain number increased by 13 equals 5 less than seven times the number. Find the number.
2. Three times a certain number is as much less than 90 as twice the number is greater than 15. Find the number.
3. The sum of two numbers is 23 and their difference is 9. Find the numbers.
4. The difference between two numbers is 27, and their sum is 5. Find the numbers.
5. The sum of three numbers is 61. The first is 7 less than twice the second, and the third is 4 less than the second. Find the numbers.
6. The perimeter of a triangle is 94 inches. The second side is 3 inches longer than twice the first, and the third side is 5 inches less than the second. What is the length of each side?
7. A rectangular garden plot requires 400 yards of fencing to inclose it. Its length is 5 yards more than twice its width. Find the dimensions.
8. Find four consecutive odd numbers whose sum is 120.
9. Find five consecutive odd numbers whose sum is 85.
10. From the sales of pop corn at a school social three pupils realized \$3.30. John collected 45 cents less than Mary, and James collected 75 cents more than John. How much did each collect?

CHAPTER VIII

PARENTHESES

Exercises

Remove the parentheses and collect like terms:

1. $7 - (3x - 4) + (5 - 6x) - (8 - 9x).$
2. $(3m - 7) - (16 + 5m) - (13m - 4).$
3. $(6a - 3b) - (7b - 10a + 8) + (14 - 9a).$
4. $4c - (6b - 2a + 10) + (8a - 12c) - 5b.$
5. $4y + (6x - 5y) - [7 + (2y - x)] + 2x - (6x + 4).$

6. $(12 ar - 15) - (12 - 18 ar) + (20 - 10 ar)$.
7. $(7 cd - 10 c) - (40 cd - 15 c) + (9 cd - 12 c + 6)$.
8. $[(5 k + 2 l) - (7 k - 4 l)] - (3 l - 2 k + 7)$.
9. $[3 r - 5 s - (2 s - 7 r)] - (8 r + 6 s)$.
10. $(2 p + 3 q) + [4 p - (2 p - 3 q) - 6 p]$.
11. $7 ax - 13 by + [(9 ax + 2 by) - (5 by - 4 ax)]$.
12. $(10 m - 9 n) - [8 n - 10 - (9 m + 6 n)] - 12$.
13. $9 - (6 u - 2 v) - [(11 - 7 u) + (8 u - 3 v)]$.
14. $[(3 a^2 - b^2) + 2 - (7 a^2 - 6)] + 4 b^2 - (5 - 6 b^2)$.
15. $(11 x^2 - 3 x) - [(4 x^2 + 5) - (3 x - 2 x^2)]$.
16. $(2 c - 7 d) + (6 c - 4 d) - (15 d + 8 c)$.
17. $(10 g - 12 h) + [(6 h - 18 g) - (8 h - 10 g)]$.
18. $3 a^2 - \left(\frac{9 ab}{2} + 4 ab\right) + \left(\frac{8 a^3}{3} - \frac{5 ab}{2}\right) + \frac{7 a^2}{3}$.
19. $(5 r^2 + 6 s) - \left[\left(3 r^2 - \frac{rs}{2}\right) + \left(7 \frac{rs}{2} - 4 s\right)\right]$.
20. $-[-5 xy + (2 x^2 + 15 x - 18) + (3 x^2 - 4 xy)]$.

In Exercises 21-25, inclose in a parenthesis preceded by a plus sign all terms containing a power of x or y , and in a parenthesis preceded by a minus sign all other terms.

21. $4 x^2 + 6 ab - a^2 - 4 xy^2 - 9 b^2 + y^2$.
22. $9 x^6 y^2 - 3 a^2 + 12 x^3 y^2 + 4 y^2 - 6 ab - 3 b^2$.
23. $-25 a^4 - 24 xy + 10 a^2 b^3 + 64 x^2 + 9 y^2 - b^6$.
24. $16 x^6 - 9 k^2 + 30 kg^4 - 40 x^3 y^2 - 25 g^8 + 25 y^4$.
25. $36 x^2 y^2 - 16 r^2 s^2 - 1 - 12 x^2 y + x^2 + 8 rs$.

In Exercises 26-30, inclose in a parenthesis preceded by a plus sign all like powers of x , and in a parenthesis preceded by a minus sign all like powers of y .

26. $2 x^2 - 3 x^3 + bx - cx^2 - 5 x - ax^3$.
27. $11 x - 2 bx^2 - x^3 + 13 x^2 + 7 dx^3 + 3 bx$.
28. $6 my^2 + 11 y - 14 ny^3 - 7 ly^2 + 9 y^3$.
29. $a^2 x - 4 aby - x + b^2 x^2 - 5 y^2 + 9 ay + 6 abx^2$.
30. $7 - c^2 y + 3 cdx - 8 dx^2 - 9 y + c^2 d^2 x^2$.

CHAPTER IX

MULTIPLICATION

Exercises

Perform the indicated multiplication :

1. $(2x + 5)(3x - 6)$.
2. $(7a - 9)(4a + 11)$.
3. $(8s - 17)(10s - 7)$.
4. $(10 - 9r)(r - 16)$.
5. $(11x - 3a)(6x + 13a)$.
6. $(8y - 15v)(11y + 5v)$.
7. $(9m + 14n)(6n - 12m)$.
8. $(7c^2 - 10c)(26c + 3c^2)$.
9. $(11x^2 - 21x)(5x - 9x^2)$.
10. $(9m^2n^2 + 11mn)(12mn - 6)$.
11. $(x^2 - 2x - 4)(2x - 5)$.
12. $(4l^2 - 3l - 2)(6l - 7)$.
13. $(2q - 9)(3q^2 + 6q - 9)$.
14. $(5n + 2n^2 - 12)(5 - 6n)$.
15. $(7y^2 - 8y - 4)(3y^2 - 9)$.
16. $(3u - 5u^2 + 10)(8u^2 + 9)$.
17. $(11 - u^2 + 6u)(5 - 3u^2)$.
18. $(z + 3z^2 + 13)(7 - 4z^2)$.
19. $(2r^2 - 9r + 5)(3r^2 + 2r)$.
20. $(7g^2 - 5gh + 3h^2)(6g - 2h)$.
21. $(9x^3 + 7x^2 - 4)(3x^2 - 5x)$.
22. $(14y - 11y^3 + 9y^2)(2y - y^2)$.
23. $(6a^3 + 8a^2 - a)(5a^3 + 4a^2)$.
24. $(x^2 + 2x - 1)(x^2 - 2x + 1)$.
25. $(2m^2 - 4m + 5)(2m^2 + 4m + 5)$.
26. $(3t^2 - 4t - 5)(t^2 + 8t + 7)$.
27. $(6l - 4l^2 + 3)(5l^2 + 2l - 9)$.
28. $(2x^2 + 2ax - a^2)(2x^2 - 2ax + a^2)$.
29. $(5h^2 + 4hk - 3k^2)(2h^2 - 9k^2 + 11hk)$.
30. $(n^2 + 2m^2 + 3mn)(n^2 - 3mn + 2m^2)$.
31. $(6a^3 + 4a - 1)(2a - 3a^3 + 2a^2)$.

32. $(2b^2 + 3m^3 - bm)(2m^3 - 3bm^2 + b^2)$.
33. $(5 - r^2 + 2r^3)(3r - cr + 2)$.
34. $(7pn + 6q^3 - 2p^2n)(3 - 4pn + 2q^3)$.
35. $(8x - 10x^2 + 5x^3)(4x^2 - 6x^3 + 3x)$.
36. $(2ax + 3a^3 - 4c)(6a^3 + 4a^2 - 5x)$.
37. $(1 - 3by + 2y^3)(2y^2 - 13by + 9y^3)$.
38. $(2x - y + 3z)(4z - 2x + 7y)$.
39. $(9k - 4g + 6p)(7g + 5k - 4p)$.
40. $(3u^3 + 2u^2v^2 - 4v^2)(3u^2v^2 - 4u^3 + 6v^3)$.
41. $(4l^2 - 10l^2m + 9m^2)(7l^3 + 8m^3 - 3lm)$.
42. $(5r^2 - 2r + 4)(7r^3 - 8r^2 + 10r)$.
43. $(3x - 7y - 5)^2$.
44. $(4k^2 - 9k - 4a + 2)(7k - 2a + 5k^2 - 6)$.
45. $(4bn^2 - 9b^2n^2 + 6n^3)(4bn^2 - 9b^2n^2 - 6n^3)$.
46. $(5x^2 - 3x - 4)(7x^3 - 5x^2 + x)$.
47. $(7y^3 - 2y^2 + 3)^2$.
48. $[5a - (3b - 2c)][9a + (3b - 2c)]$.
49. $(5g^3 - 7gh^2 - 9h^3)(9h^3 - 11gh^2 + g^3)$.
50. $(p^3 - 2pq + q^2)(p + q)^2$.
51. $[2m - 5n - (2l - 3k)][2m - 5n + (2l - 3k)]$.
52. $(3r^2s - 5rs)(rs - 2rs^2)(r^2s^2 + 4rs)$.
53. $(5x^2 - 7xy + 4y^2)(3x - 4x^2 + 6xy - 2)$.
54. $(4b^3 - 2b^2)(9b + 3b^3)(8b^2 - 4b)$.
55. $(3x^2 - 2x^2 + x)(5x^3 + x^2 - 3x + 4)$.

Problems

1. Find the area and the perimeter of a rectangle whose length is $(7x - 3y)$ inches, and whose altitude is $(2x + y)$ inches.

2. From each corner of a square piece of tin of side m inches, a square of side n inches is cut. By turning up the sides an open box is formed. Show that $m^2 - 4n^2$ square inches is the area of the inside of the box.

3. Express the area of Problem 2 as the product of two binomials.

4. Using the results of Problem 3, find by a short method the area of the inside of the box if $m = 13$, $n = 3$; if $m = 82$, $n = 6$.

5. The dimensions of a rectangular box are w , $w - 2$, and $w + 3$. Express (a) the sum of the edges, (b) the total outside surface, and (c) the volume of the box.

6. For a cube whose edge is $(3h - 2k)$ find (a) the sum of the edges, (b) the total outside surface, and (c) the volume.

7. If two equal rectangular boxes of dimensions l , $l - 2$, and $l - 3$ are placed end to end, find (a) the sum of the outer edges, (b) the outer surface, and (c) the combined volume.

8. The formula for finding the area of a circle is πr^2 , in which $\pi = \frac{22}{7}$ and r equals the radius of the circle. What is the area of a circular ring made by cutting a circle of radius x from the center of a circle of radius y ?

9. If $x = 6$ inches and $y = 8$ inches in Problem 8, express the area of the resulting ring.

10. Express the area of a circle whose radius is $(2a - 3b)$ inches.

11. The total surface of a cylinder is $2\pi rh + 2\pi r^2$, in which r is the radius of the base and h is the height of the cylinder. Find the amount of tin in a cylindrical can which has a circular base with a radius of 7 inches and a height of 10 inches if $\pi = \frac{22}{7}$.

12. Express the total surface of a cylinder the radius of whose base is $(r - 3)$ and whose height is $(r + 2)$.

13. The formula for finding the volume of a cylinder is $\pi r^2 h$, in which r is the radius of the base and h the height of the cylinder. Find the cubic contents of 100 cans whose dimensions are given in Problem 11.

14. Express the volume of a cylinder whose height is $(2r + 2)$ and the radius of whose base is $(r - 4)$.

15. The formula for finding the surface of a sphere is $4\pi r^2$, in which r is the radius of the sphere. Compare the sum of the surfaces of 630 balls of $\frac{1}{2}$ -inch radius with the surface of a ball having a radius of 7 inches.

16. Compare the surface of a sphere of radius $2a$ with the surface of a sphere of radius a .

17. The formula for finding the volume of a sphere is $\frac{4}{3}\pi r^3$. How many spherical balls of $\frac{1}{2}$ -inch radius can be made from a spherical ball of lead having a radius of 3 inches?

18. State the formula for finding the volume of a sphere whose radius is $(2r - 1)$.

19. Compare the volumes of the two spheres of Problem 16.

20. Compare the volume of a sphere of radius k with that of a sphere of radius l .

CHAPTER X

PARENTHESES IN EQUATIONS

Exercises

Solve for the unknown:

1. $5(2x - 4) = 6(x + 8)$.
2. $3(4r - 5) + 11 = 8(3r + 7)$.
3. $7 + 2(5t - 9) = 4(5t + 6) - 13$.
4. $10 - 3(2 - 3n) = 22 - (n - 12)$.
5. $18 + 2(21 - 5s) - 8(3s + 11) = 23$.
6. $7h - 4(6 - 11h) = 5(9h + 12) - 8h$.
7. $6(5c - 4) - 13c - 2(10c + 18) = 9$.
8. $3(4y - 11) - 4(15 + 5y) + 7(8y - 9) = 0$.
9. $7(5w - 10) + (10 - 30w) - 40 = 9(16 + 4w) - 7w$.
10. $2(v - 18) - 5(2 - 11v) = 8(6 + 5v) - 8(15 - 7v)$.
11. $4(2 - 5z) + 6(z - 16) - 7(12 - 9z) = 8(5z - 14)$.
12. $6(12 - 10p) - 5(8p + 9) = 2(13p - 19) + 8(11 - 7p)$.
13. $(2x - 5)^2 + 13 = 10 + 4(x - 7)^2$.
14. $(4 - t)(14 + t) - 8 = (t + 17)(2 - t) - 13t$.
15. $(2m - 8)^2 + 11m = (8 - 4m)(12 - m) + 38$.
16. $17 - 3(4 - k)^2 = 14k - (10 + k)(3k + 7) - 8$.
17. $(5 - 6s)(s + 4) = 7s - 3(2s + 10)(s - 14)$.
18. $4(2 - 5b)(b + 7) + 14b = 32 + (19 - 2b)(10b + 6)$.

Problems
Problems

1. The sum of two numbers is 66. Seven times the one equals four times the other. What are the numbers?

② Find three consecutive even numbers such that the product of the first two is 112 less than the square of the third.

3. Find four consecutive odd numbers such that the product of the first and second is 176 less than the product of the third and fourth.

4. A square has the same area as a rectangle whose length is 18 inches greater and whose breadth is 6 inches less than the side of the square. Find the dimensions of each.

5. A room is 7 feet longer than it is wide. If the length is increased by 4 feet and the width is decreased by 5 feet, the area is decreased by 70 square feet. Find the original dimensions.

6. A rectangular field is 30 rods longer than it is wide. The perimeter of the field is 260 rods. Find its dimensions.

7. A rectangular lot is 42 yards longer than it is wide. If the width is decreased by 5 yards and the length is decreased by 8 yards, the area is decreased by 638 square yards. What are the original dimensions?

⑧ The value of 27 coins consisting of dimes and nickels is \$2.15. Find the number of each.

9. A collection of 30 coins consisting of nickels, dimes, and quarters in which there are twice as many dimes as nickels, has a value of \$4.00. Find the number of each.

10. A is three times as old as B. Six years ago A was twice as old as B will be four years from now. What are their present ages?

11. A is 57 years old and B is 23 years old. In how many years will A be only twice as old as B?

12. A is 18 years old and B is 42 years old. How many years ago was B three times as old as A?

CHAPTER XI

DIVISION

Exercises

Perform the division as indicated:

- ① $(2p^2 + 11p - 21) \div (2p - 3).$
- ② $(6n^2 - 21n - 66) \div (3n + 6).$
- ③ $(15a^2x^2 - 11ax - 56) \div (3ax - 7).$
- ④ $(54b^2y^4 - 39by^2 - 5) \div (9by^2 + 1).$
- ⑤ $(14a^2k - 20a^3 + 27ak^2 - 18k^3) \div (3k^2 - 4a^2 - 2ak).$
- ⑥ $(15y^2 + 11y - 14) \div (5y + 7).$
- ⑦ $(28k^3 - 21k^4 + 29k^2 - 20k - 10) \div (7k^2 - 5).$
- ⑧ $(2p^3 - 7cp^2 + 5c^2p + 2c^3) \div (p - 2c).$
- ⑨ $(3d - 8du^3 - 3du^2 + 8du^5) \div (3d - 8du^3).$
- ⑩ $(66a^2s^2 + 25s^4 + 49a^4) \div (5s^2 - 2as + 7a^2).$
- ⑪ $(20w^4 - 8w^3 - 39w^2 + 4w^5 + 5w) \div (w^2 + 5w).$
- ⑫ $(21ch^2 - 56h^3 - 18c^3h - 6c^2 + 48c^2h^2 + 16ch) \div (3c - 8h).$
13. $(9y^4 - 16a^2y^2 + 48a^3y - 36a^4) \div (3y^2 - 4ay + 6a^2).$
- ⑭ $(69z^2 - 21z^3 - 95z + 22) \div (2 - 7z).$
- ⑮ $(2x - 3x^2 + 3x^3 + 54) \div (3x + 6).$
16. $(m^2 + n^2 + p^2 + 2mn + 2mp + 2np) \div (m + n + p).$
- ⑰ $(9h^3 + h^4 - 30h + 2h^2 - 25) \div (h^2 - 5).$
- ⑱ $(acx + bdy - bcy - adx) \div (ax - by).$
19. $(-6a^4 + 27a^3b - 44a^2b^2 + 35ab^3) \div (3a^2 - 6ab + 7b^2).$
- ⑳ $(cp^3 - 8c) \div (cp^2 + 2cp + 4c).$
- ㉑ $(-acw - d^2w^2 + cdw^2 + adw) \div (aw - dw^2).$
- ㉒ $(15p^4 - 13p^3 - 92p^2 + 72p - 152) \div (5p^2 - p - 38).$
23. $(2u^5 - 23u^2 - 5u^3 + 12u^4 + 42u) \div (u^2 + 6u).$

24. $(10 h^2 k^2 + 10 h^5 - 8 h^3 k - 5 h^4 k - 27 h k^3 + 30 h^3 k^2 + 18 k^4) \div (-4 h k + 5 h^3 + 3 k^2)$.
25. $(-70 g^2 + 6 g^4 + 13 g^3 + 71 g - 20) \div (4 + 3 g^2 - 7 g)$.
26. $(16 r^4 + s^4 + 4 r^2 s^2) \div (4 r^2 + s^2 - 2 r s)$.
27. $(37 z^2 + z^5 - 60 z + 57) \div (z^2 + 10 - 2 z)$.
28. $(h^2 - k^2 - l^2 + 2 k l) \div (h + k - l)$.
29. $(p^4 + p^8 - 3 p^6 - 7 p^2) \div (1 - p^2)$.
30. $(q^6 + 4 d^4 - 4 d^2 q^3 - 9 d^2 q^2) \div (q^3 - 3 d q - 2 d^2)$.
31. $(16 b^4 h^8 - b^8) \div (2 b h^2 + b^2)$.
32. $(32 p^{10} + c^5) \div (c + 2 p^2)$.
33. $(a^8 n^4 - 16 a^4) \div (a^2 n - 2 a)$.
34. $(14 g^3 h + 24 g h - 8 g^2 h^2 - 17 g^2 - 23) \div (2 g h - 3)$.
35. $(u^2 - 9 b^2 v^2 + 6 b v s - s^2) \div (u - 3 b v + s)$.
36. $(r^5 - r s^4 - r^4 s + s^5) \div (-2 r s + r^2 + s^2)$.
37. $(p^2 + q^2 - r^2 - 2 p q) \div (p - q - r)$.
38. $(u^4 + u^2 v^2 + v^4) \div (v^2 - u v + u^2)$.
39. $(c^6 p^6 - 64 s^6) \div (c p - 2 s)$.
40. $(3 x^3 - 6 - 5 x - 2 x^2) \div (1 + x)$.
41. $1 \div (1 - a)$.
42. $(4 u^4 + w^4 - 12 u^2 v + 4 u^2 w^2 + 9 v^2 - 6 v w^2) \div (2 u^2 - 3 v + w^2)$.

CHAPTER XII

EQUATIONS AND PROBLEMS

Exercises

Solve for x :

1. $100 - 14 x + 36 = 18 - 6 x + 22$.
2. $(2 x - 9)(2 x - 2) = (2 x - 3)(2 x - 7) - 1$.
3. $(8 x + 1)(10 x + 1) + 18 = (4 x + 1)(20 x + 1)$.
4. $x + 2 m - 8 = 2 x + 3 m - 40$.
5. $2(x - c) + 2 a = 4 x + 2(c - 2 a)$.
6. $2 m^2 + 2 m x - x^2 = (m + x)(m - x)$.

7. $(x + a)(x - 3a) = (x + 2a)^2$.
8. $(x - m)^2 + (m - n)^2 = (x + m)^2 + (m + n)^2$.
9. $(x - m - n)^2 - (x - m)(x - n) = 0$.
10. $4x + 2a + m(6 + 3a) = 3mx + 2(3a + 4)$.
11. $m(2m + 3c) - (x + m)(x + c) + x^2 = 0$.
12. $n^3 + 2x = n + nx + 6$.
13. $nx + x = n^3 + 2n + 3$.
14. $nx + 4a^2n = n^3 + ax + 3a^3$.
15. $n^3 - 6 = 11n + 3x + nx$.
16. $n^4 + 3x + 2n + 12 = 11n^2 + nx$.
17. $n^3 + 2a^3 + 2ax = 5a^2n + nx$.
18. $nx + n^3 - 6m^3 = 3mx + 7m^2n$.
19. $4n^4 + x = nx - 3n + 7$.
20. $3n^3 + 2na^2 - 12a^3 + 2ax = 4n^2a + nx$.
21. $9m(m + c) - (x + 3m)(x - 3m) = (3m - x)(x - 3c)$.
22. $10n^2 - 3(2m^2 + 3nx) - m(2x - n) = (5m - 4n)x - 5(n^2 + 4m^2 + 6mn)$.
23. $(x - 2a + 3m)^2 = 6am - (2a - x)(x + 3m)$.
24. $a^2x + ax + 3c^2 + x = 3a^3c^2$.
25. $n^2x + 2n^2 + 5x + 10 = n^3 + 5n$.
26. $x + 16r^2x = 3n^2 + 192n^2r^3 + 4rx$.
27. $n^2x - 9(3a^2 - x) = a^2n^3 + 3nx$.
28. $x + cx = c^4 - c^2x + c^2 + 1$.
29. $a^4 + 2ax + 16 + 4a^2 = a^2x + 4x$.
30. $a^4 + 2ax + 4 = a^2x + 2x$.
31. $4a^4 + ax + 3x = 2a^2x + 13a^2 - 9$.

Problems

1. Two men start at the same time from the same place and travel in opposite directions. One travels 5 miles per hour faster than the other. In 10 hours they are 500 miles apart. Find the speed of each.

2. Two men start at the same place and the same time and travel in the same direction. The first travels 6 miles per hour more than the second, but is delayed 3 hours on the way. At the end of 10 hours they are 24 miles apart. Find the speed of each.

3. In playing a game each player was directed to add 18 to his age, to multiply the result by 4, to subtract 72 from the product, and then add twice his age. When the leader in the game was given the final result he at once gave the player's age. How did he do it?

4. The circumference of the front wheel of a wagon is 8 feet and that of the rear wheel 10 feet. How far has the wagon gone when the front wheel has made 50 revolutions more than the rear wheel?

5. In Problem 4, how many less revolutions will the rear wheel make than the front wheel when the wagon goes one mile?

6. A and B together have 100 dollars. A gives n dollars to B, after which they have equal amounts. How many dollars had each at first?

7. A and B have d dollars. B gives n dollars to A, after which they have equal amounts. Find the number of dollars each had at first.

8. A and B have d dollars. A gives 10 dollars to B. Then B gives n dollars to A, after which each has the same amount. Find the number of dollars each had at first.

9. The altitude of a rectangle is a and its base is b . If the altitude is decreased 2 feet, how much must the base be increased so that the area will be the same as before?

10. If in Problem 9 the altitude is increased m feet, how much must the base be decreased if the area is to remain unchanged?

11. The altitude, a , of a rectangle is decreased m feet. Find the increase in the base, b , which will leave the area unchanged.

CHAPTER XV

SOLUTION OF EQUATIONS BY FACTORING

Exercises

Solve for x by factoring:

1. $x^2 - 15x + 50 = 0$.
2. $x^2 - 3ax - 4a^2 = 0$.
3. $x^2 - 18 = 7$.
4. $x^4 - 20x^2 + 64 = 0$.
5. $x^4 + 36a^4 = 13a^2x^2$.
6. $3x^2 - 5x + 2 = 0$.
7. $2x^2 - 7ax + 5a^2 = 0$.
8. $2x^2 - 2x - 12 = 0$.
9. $x^2 - 3ax + 3bx = 9ab$.
10. $2x^2 - ax = 2am - 4mx$.
11. $x^3 + 4a^2m = x(4a^2 + x^2)$.
12. $m^2(x - c) = 16m(mx + m)$.
13. $ax^2 + 3bx - 2abx = 6b^2$.
14. $2x^2 + 5x + 3 = 0$.
15. $3x^2 - 5ax + 2a^2 = 0$.
16. $5x^2 - 8ax + 3a^2 = 0$.
17. $6x^2 = ax + 12a^2$.
18. $(3x - 2a)(x - a) - (x - a) = 0$.
19. $(2x + 5a)(x - 2a) = x - 2a$.
20. $(3x - a)(x - a) - (2x - a)(x - a) = 0$.
21. $(a - 3x)(x + 2a) - (x + 2a)(4a - x) = 0$.
22. $(x - m)(m + x) - (x + 8m)(x - m) = 0$.
23. $(x + 5m)(x - 5m) - (3x - 4m)(x + 5m) = 0$.
24. $x^2 - a^2 + cx - ac = 0$.
25. $x^3 + 2mx^2 = 3a^2x + 6a^2m$.
26. $m^2x^2 + x^2 = m^2x + x$.
27. $m^2 + mx^2 = m^2x + x^2$.
28. $2ax^2 + x^2 = 2a^2x + a^2$.
29. $3ax^2 + 2a^2 = 3a^2x + 2x^2$.
30. $a^2x^2 + ax^2 + x^2 = a^2x^2 + a^2x + a^2$.
31. $a^2x^2 + a^2x + a^2 = a^2x^2 - ax^2 + x^2$.
32. $3m^2 + mx^2 = m^3x + 3x$.
33. $a^2(x^2 - 2x + 4) - x^2(a^2 - 2a + 4) = 0$.
34. $x^2(a^2 - am + m^2) = a^2(x^2 - mx + m^2)$.
35. $amx^2 - anx + mcx = nc$.

Problems

1. The product of two consecutive numbers is 306. Find the numbers.
2. The product of two consecutive odd numbers is 399. Find the numbers.
3. The digits of a two-digit number are consecutive integers, and the number is 36 greater than the sum of the digits. Find the number.
4. A number of two consecutive odd digits is 5 greater than two times the product of the digits. Find the number.
5. The units' digit of a two-digit number is 3 greater than the tens' digit, and the product of the digits is 19 less than the number. Find the number.
6. The altitude of a triangle is 3 greater than the base. If its altitude were increased 2 and its base decreased 2, the area would be 30. Find the base and altitude.
7. The altitude of a triangle is twice the base. If the base is increased by m and the altitude decreased by m , the area is increased by m^2 square units. Find the altitude and the base.
8. The altitude of a triangle is a feet and the base is b feet. Were the altitude increased by m feet and the base decreased by $2m$ feet the area would be unchanged. Find m .
9. The altitude of a rectangle is a and the base is b . The altitude is increased and the base decreased by the same amount, leaving the area unchanged. Find the change in the base and altitude.
10. If $a = 10$ feet and $b = 12$ feet in Problem 8, find m .
11. The base and the altitude of a triangle are consecutive even numbers. If each is increased by 4, the area is 92 square units greater. Find the base and the altitude.
12. The base of a triangle is p feet and the altitude is q feet. If the base is decreased $2m$ feet and the altitude increased $2m$ feet, the area will be $mp - 3m^2$ square feet greater than before. Find m .
13. The altitude of a triangle is 6 feet less than the base. If the altitude is increased m feet and the base decreased m feet, the area will be $m + 2$ square feet greater than before. Find m .

14. The altitude of a rectangle is 2 feet less than the base. If the base is decreased m feet and the altitude increased m feet the area will be $2m + 5$ square feet less than before. Find m .

15. In Problem 12, if $p = 42$ and $q = 18$, find the new area.

16. A man bought a number of oranges for \$1.20. Had each cost 2 cents less he would have received 5 more for the same sum. Find the number bought and the price of each.

17. A number of bananas cost 96 cents. Had they each cost 2 cents more the number received would have been 8 fewer. Find the number bought and the price of each.

18. A number of pounds of coffee were bought for \$18. At 10 cents a pound less, twice as many pounds could have been bought for \$4 more. Find the number of pounds and the price per pound.

CHAPTER XVI

FRACTIONS

Exercises

Reduce to lowest terms:

1. $\frac{x-1}{mx^2n-mn}$
2. $\frac{x^2-2x+1}{x^2-4x+3}$
3. $\frac{a^2-2a-15}{a^2-14a-51}$
4. $\frac{n^2-12n+35}{n^2-10n+25}$
5. $\frac{70(x^2-a^2)}{21(x-a)^2}$
6. $\frac{2n^2-2n-12}{4n^2-16}$
7. $\frac{n^2-12n+36}{n^3-8n^2+12n}$
8. $\frac{n^2+n-2}{2n^2-3n+1}$
9. $\frac{2n^2+19n+35}{2n^2-n-15}$
10. $\frac{3n^2-8an-3a^2}{3n^2-5an-2a^2}$
11. $\frac{12v^2-28v+8}{6v^2+28v-10}$
12. $\frac{6m^2-13am-15a^2}{6m^2-am-5a^2}$
13. $\frac{2an-2n+2x-2an}{ax-2n-an+2x}$
14. $\frac{a^2-b^2-2bc-c^2}{a^2+2ab+b^2-c^2}$
15. $\frac{(n-a)^2-(b+c)^2}{n^2-(a+b+c)^2}$
16. $\frac{n^4-n^3x-n^2x^2+nx^3}{n^5-n^4x-nx^4+x^5}$

Reduce to a fraction :

$$17. x - y - \frac{x^2}{x + y}.$$

$$20. a^2 - \frac{a^3 + b^3}{a - b} + ab + b^2.$$

$$18. x - 3 + \frac{x^2 - 5x}{x + 3}.$$

$$21. x^2 - xy + y^2 - \frac{x^3}{x + y}.$$

$$19. a - b - \frac{b^2 + a^2}{a + b}.$$

$$22. a^2 + c^2 - \frac{a^4 + c^4}{a^2 - ac + c^2} + ac.$$

$$23. x + y - \frac{x^2 + y^2 - 2xy - z^2}{x + y - z} + z.$$

Reduce to a mixed number :

$$24. \frac{x^2 + 5x + 6}{x + 4}.$$

HINT. Dividing numerator by denominator gives

$$x + 1 + \frac{2}{x + 4}.$$

$$25. \frac{x^2 - 7x + 10}{x + 1}.$$

$$28. \frac{x^3 - x + 1}{x + 2}.$$

$$31. \frac{x^3 - 8}{x + 2}.$$

$$26. \frac{x^2 + 8x - 4}{x - 3}.$$

$$29. \frac{x^3 - 5x^2 + 6}{x + 5}.$$

$$32. \frac{x^2 + y^2}{x - y}.$$

$$27. \frac{3x^2 + 5x - 3}{x + 6}.$$

$$30. \frac{x^2 + 9}{x + 3}.$$

$$33. \frac{x^3 - y^3}{x + y}.$$

$$34. \frac{x^4 - 16}{x + 1}.$$

$$35. \frac{x^4 + x^2a^2 - a^4}{x^2 + x^2a^2 + a^2}.$$

Perform the indicated additions and subtractions :

$$36. \frac{1}{ac} + \frac{1}{ab} - \frac{1}{bc}.$$

$$40. \frac{m}{x} + \frac{x}{m} - \frac{a}{c}.$$

$$37. \frac{a}{bc} - \frac{b}{ac} + \frac{c}{ab}.$$

$$41. \frac{a}{n} - \frac{n}{a} + \frac{2 - n}{a}.$$

$$38. \frac{5n - 2}{8} - \frac{2n + 5}{4} - \frac{3n - 7}{7}.$$

$$42. \frac{1}{a + x} + \frac{2}{a - x}.$$

$$39. \frac{x - 3}{2} - \frac{3x - 1}{3} - \frac{5x + 4}{6}.$$

$$43. \frac{a + n}{a - n} + 2 + \frac{a - n}{a + n}.$$

$$44. \frac{1}{x} - \frac{3x-y}{x^2+xy}.$$

$$48. \frac{x+3}{x-3} - \frac{x-3}{x+3} + 1.$$

$$45. \frac{a}{a-n} + \frac{a}{n} + \frac{n}{a}.$$

$$49. \frac{1}{x^2-9} - \frac{2}{x^2-5x+6}.$$

$$46. m - \frac{m^2+a^2}{m+a} - a.$$

$$50. \frac{a}{mn} + \frac{b}{mr} + \frac{c}{nr} + 2.$$

$$47. a^2 + \frac{a^2n^2-n^4}{a^2+an+n^2} - an + n^2.$$

$$51. \frac{a-n}{an} + \frac{n-x}{nx} + \frac{a-x}{ax}.$$

$$52. \frac{5m+2}{m^2-4} - \frac{3}{m-2} + \frac{4}{m+2}.$$

$$53. \frac{1}{a^2-3a+2} - \frac{2}{a^2+2a-3} - \frac{3}{a^2+a-6}.$$

$$54. \frac{a^2}{a+1} + \left(\frac{a^2}{2(a+1)} \right)^2.$$

$$58. \frac{a^2}{2a+2} \pm \frac{a(a+2)}{2(a+1)}.$$

$$55. \left(\frac{a^2}{2(a-3)} \right)^2 - \frac{3a^2}{a-3}.$$

$$59. \frac{a^2}{2a-6} \pm \frac{a^2-3a}{2(a-3)}.$$

$$56. -\frac{2m^2}{3m-2} + \left(\frac{-3m^2}{2(3m-2)} \right)^2.$$

$$60. \frac{-2m^2}{6m-4} \pm \frac{3m^2-4m}{2(3m-2)}.$$

$$57. \left[\frac{-5a^2}{2(5a-7n)} \right]^2 - \frac{7a^2n}{5a-7n}.$$

$$61. \frac{5a^2}{10a-14n} \pm \frac{5a^2-14an}{2(5a-7n)}.$$

$$62. \frac{m^2+mn+n^2}{m+n} + \frac{m^2-mn+n^2}{m-n}.$$

$$63. \frac{a-x}{a+x} - \frac{a^2+ax+x^2}{a^2-x^2} + 3.$$

$$64. \frac{1}{x-1} - \frac{2}{x-2} + \frac{3}{(2-x)(x-1)}.$$

$$65. \frac{1}{(a-x)(a-n)} - \frac{3}{(x-n)(a-x)} + \frac{2}{(n-a)(n-x)}.$$

$$66. \frac{x+3}{x^2-5x+6} - \frac{x+4}{9-x^2} + \frac{2}{3-x}.$$

$$67. \frac{a}{a-b} + \frac{b}{b-a} + \frac{c}{c-a} - \frac{a+b+c}{c-b}.$$

$$68. \frac{x-m-n}{x+m+n} - \frac{x-m+n}{x+m-n} + \frac{4xn}{(x+m)^2-n^2}.$$

$$69. \frac{m+1}{(m-n)(m-x)} + \frac{n+1}{(n-m)(n-x)} - \frac{x+1}{(m-x)(x-n)}.$$

$$70. \frac{4m^2-9}{2mn} - \frac{6-m}{5m^2} - \frac{m^2-4}{mn^2} + \frac{45mn+12n^2+10m^3}{10m^2n^2}.$$

Perform the indicated multiplications and divisions:

$$71. \frac{2ax}{3by} \cdot \frac{3ay^2}{10bc} \cdot \frac{20b}{12a^2x^2}.$$

$$74. \frac{n^2-1}{m^2+m} \left(1 + \frac{m}{1-m} \right) \frac{m^2-1}{n+1}.$$

$$72. \frac{3a^2c}{4an^2} \cdot \frac{5c^2n}{6ac} \cdot \frac{18a^2}{10ac^2}.$$

$$75. \frac{c^2-2ac}{2a-c} \cdot \frac{4a}{6a-2c}.$$

$$73. \frac{a}{c} \left(\frac{a+c}{ac} \right) \cdot \frac{c^2}{a(a-c)}.$$

$$76. \left(\frac{m}{n} + \frac{n}{m} \right) \cdot \left(m - \frac{n^2}{n} \right).$$

$$77. \frac{v^2-3v}{4v^2-12v} \cdot \frac{3v^2-12v}{v^2+4v}.$$

$$78. \frac{a^2+2a}{a^2-4} \cdot \frac{3a^2+15a}{a^2-25} \cdot \frac{a^2-7a+10}{3}.$$

$$79. \left(\frac{n^2}{m^2} - 1 \right) \left(\frac{m-n}{m+n} + 1 \right).$$

$$80. \frac{m^2}{n^2-m^2} \left(1 - \frac{n}{m} \right) \left(1 + \frac{m}{n} \right).$$

$$81. \frac{6m}{a} \left(\frac{9m^2}{4a} \right)^2 \left(\frac{-2a}{3m} \right)^3 \cdot \frac{1}{9m^2}.$$

$$82. \frac{3m}{9-m^2} \cdot \frac{m^2-5m+6}{18mn} \cdot \frac{18n-6mn}{am+2a}.$$

$$83. \frac{4m^a-2n^a}{(2m^a-n^a)^2} \cdot \frac{4m^{2a}-n^{2a}}{(2m^a+n^a)^2} \cdot \frac{2m^a+n^a}{3}.$$

$$84. \frac{8m-24n}{m^2-6mn+9n^2} \cdot \left(4m - \frac{n^2}{m} \right) \left(\frac{4m^2+n^2}{64m^4-4n^4} \right) \cdot \frac{3m^2-9mn}{2a+2x}.$$

$$85. \frac{a^4-n^4}{(a-n)^2} \div \frac{a+n}{a-n}.$$

$$86. \frac{m^2-2mn+n^2}{m^4-n^4} \div \frac{m-n}{m^2+mn}.$$

$$87. \frac{a^2 + 4a - 12}{a^2 + 11a + 30} \div \frac{a^2 + 8a + 15}{a^2 + a - 6}.$$

$$88. \left(\frac{m}{m-n} - \frac{n}{m+n} \right) \div \left(\frac{m}{m+n} - \frac{n}{m-n} \right).$$

$$89. \left(3 - \frac{a-5x}{a-2x} \right) \div \left(5 - \frac{a^2-19x^2}{a^2-4x^2} \right).$$

$$90. \left(1 + \frac{2}{2a} - \frac{3}{4a^2} \right) \left(\frac{72a^3}{3(4a^2-36a+17)} \right) \cdot \frac{6a-51}{8a^2+12a}.$$

$$91. \left(15 + \frac{1}{n} - \frac{2}{n^2} \right) \div \left(6 - \frac{7}{n} - \frac{3}{n^2} \right) \left(10 - \frac{19}{n} + \frac{6}{n^2} \right).$$

$$92. \left(12a - 11 - \frac{7}{2a} \right) \div \left(2 + \frac{11}{2a} + \frac{5}{4a^2} \right) \left(\frac{1}{40a^3 - 250a} \right).$$

$$93. \frac{a^2 + 2ab + b^2 - c^2}{a^2 - c^2} \div \left(\frac{a^2 - b^2 + 2bc - c^2}{a^2 - ac + ab - bc} \right) \frac{(a+b+c)(a+b)}{a^2 + 2ac + c^2 - ab - bc}.$$

$$94. \left(\frac{4n}{m} - \frac{15n^2}{m^2} + 4 \right) \div \left(4 - \frac{16n}{m} + \frac{15n^2}{m^2} \right) \left(3 - \frac{4m+20n}{2m+5n} \right).$$

$$95. \left(2 - \frac{17}{5x} - \frac{11}{5x^2} \right) \div \left(5 - \frac{23}{3x} - \frac{22}{3x^2} \right) \left(3 + \frac{7}{2x} + \frac{1}{x^2} \right).$$

$$96. \left(\frac{1}{a^2+3a+2} - \frac{1}{a+2} \right) \div \left(\frac{2}{a^2+3a+2} - \frac{1}{a+1} \right).$$

$$97. \frac{(m+n-x)^2 - (m+n+x)^2}{(m-n-x)^2 - (m-n+x)^2} \div \frac{2m-2n}{3m+3n}.$$

$$98. \frac{x^2 + m^2 + 2mx}{m^2 - 3mx - 4x^2} \cdot \frac{1}{m+2x} \cdot \left(2x+m - \frac{5(m+2x)}{x+m} \right).$$

$$99. \left(\frac{a-3x}{a+x} \right) \left(1 + \frac{4x}{a+x} \right) \div \left(\frac{4a}{x} - 7 - \frac{15x}{a} \right).$$

$$100. \left(1 + \frac{2}{a-1} \right) \left(\frac{a^2+a-2}{a^2+a} \right).$$

$$101. \left(\frac{a^2}{x^2} + \frac{a}{x} + 1 \right) \div \left(\frac{a^2}{x^2} - \frac{a}{x} + 1 \right).$$

$$102. \left(\frac{x}{y^2} - \frac{2}{y} + \frac{1}{x} \right) \div \left(1 - \frac{x}{y} \right).$$

$$103. \left(\frac{1}{a^2} + \frac{1}{a} + 1 + a \right) \div \left(a + 1 - \frac{1}{a} - \frac{1}{a^2} \right).$$

$$104. \left(6m - 1 - \frac{1}{m} \right) \div \frac{2m - 1}{3a}.$$

$$105. \left(\frac{a - 5}{2} - 7 + \frac{24}{a} \right) \div \frac{9 - 3a}{a}.$$

$$106. \left(a - 4 + \frac{9}{a + 2} \right) \div \left(1 - \frac{4a - 7}{a^2 - 4} \right).$$

$$107. \left(m + \frac{3m + 6}{m^2 - 1} + 2 \right) \div \left(m + 3 + \frac{1}{m + 1} \right).$$

$$108. \left(\frac{12}{a - 3} + 1 - \frac{4}{a - 1} \right) \left(\frac{4}{a + 1} + 1 - \frac{12}{a + 3} \right).$$

COMPLEX FRACTIONS

Exercises

Simplify :

$$1. \frac{\frac{m}{n} - \frac{c}{x}}{\frac{m}{n} + \frac{c}{x}}.$$

$$5. \frac{\frac{a^2 + m^2}{2a} - m}{\frac{a}{m} - \frac{m}{a}}.$$

$$9. \frac{\frac{4a + 2}{3}}{\frac{2a + 1}{5a}}.$$

$$2. \frac{a - \frac{1}{a}}{a + \frac{1}{a}}.$$

$$6. \frac{3a - 1 - \frac{2}{a}}{\frac{2a - 2}{3a}}.$$

$$10. \frac{\frac{n}{x} + \frac{x}{m} - \frac{m}{n}}{\frac{1}{m} + \frac{1}{x} - \frac{1}{n}}.$$

$$3. \frac{x + 1 - \frac{1}{x - 1}}{x - 1 - \frac{1}{x + 1}}.$$

$$7. \frac{\frac{2m - 5}{2} + \frac{21}{m} - 7}{3 - \frac{18}{m}}.$$

$$11. \frac{4 - \frac{3}{m + 1}}{6 + \frac{5}{m^2 - 1}}.$$

$$4. \frac{\frac{m + n}{n} - \frac{m + n}{m}}{\frac{1}{n} - \frac{1}{m}}.$$

$$8. \frac{\frac{1}{m + 1}}{1 - \frac{1}{m + 1}}.$$

$$12. \frac{5a - 2 - \frac{14}{5a + 3}}{5a - 1 - \frac{21}{5a + 3}}.$$

$$13. \frac{\frac{1}{1-m} + \frac{1}{1+n}}{\frac{1}{1-m} - \frac{1}{1+n}}$$

$$18. \frac{\frac{a}{a+n} - \frac{a}{a-n}}{\frac{a}{a-n} + \frac{a}{a+n}}$$

$$23. \frac{\frac{8c}{a} + 2 - \frac{a}{c}}{\frac{a}{c} + 1 - \frac{20c}{a}}$$

$$14. \frac{\frac{1}{mn} + \frac{1}{mx} + \frac{1}{nx}}{\frac{m^2 - (n+x)^2}{mn}}$$

$$19. \frac{\frac{a}{c} + \frac{a-c}{a+c}}{\frac{a-c}{a+c} - \frac{a}{c}}$$

$$24. \frac{\frac{a+c}{c} - \frac{a-c}{a}}{\frac{a+c}{a} + \frac{a-c}{c}}$$

$$15. \frac{\frac{9}{x} - x}{\frac{1}{x} - \frac{2}{a^2} - \frac{3}{a^3}}$$

$$20. \frac{\frac{m+n}{m-n} - \frac{m-n}{m+n}}{\frac{m-n}{m+n} - \frac{m+n}{m-n}}$$

$$25. \frac{\frac{m}{m+2} - \frac{2-m}{m}}{\frac{m}{2+m} + \frac{2-m}{m}}$$

$$16. \frac{\frac{2}{n} - \frac{3}{m}}{\frac{2m}{3n} - 2 + \frac{3n}{2m}}$$

$$21. \frac{x-2 - \frac{14}{x+3}}{\frac{21}{x+3} + 1 - x}$$

$$26. \frac{\frac{1}{a} + \frac{1}{c} + \frac{1}{x}}{\frac{ax+ac+cx}{acmx}}$$

$$17. \frac{\frac{(x+n)^2 - 4}{nx}}{\frac{x}{n} - \frac{n}{x}}$$

$$22. \frac{\frac{8n}{n-6} - n - 3}{n+5 - \frac{4n+6}{n-6}}$$

$$27. \frac{\frac{(a-6c)^2 + 2}{12ac}}{\frac{1}{2y} - \frac{3}{x}}$$

$$28. \frac{\frac{5a-4x}{4a} + 20x}{\left[\frac{(5a+4x)^2}{4a} \right]^2 - \frac{5x}{a}}$$

$$31. \frac{\left(3a+4 + \frac{4}{3a} \right) \left(\frac{3a-2}{a-1} \right)}{\frac{9a^2-4}{9a^2+9a}}$$

$$29. \frac{\frac{(m+5n)^2}{10mn} - 2}{10 - \frac{2m}{n}}$$

$$32. \frac{\frac{(3a-m)^2}{4a^2}}{\left(\frac{m-3a}{2a^3} \right)^2}$$

$$30. \frac{\frac{38x}{a} - \left(\frac{a+4x}{a} \right)^2 - 8}{8a - \frac{(3a+2x)^2}{3x}}$$

$$33. \frac{\frac{(x-6a)^2}{6ax} + 4}{\left(\frac{1}{a} + \frac{6}{x} \right)^2}$$

CHAPTER XVII

EQUATIONS CONTAINING FRACTIONS

Exercises

Solve for the unknown:

$$(1) \frac{5x}{6} - \frac{x-2}{9} = \frac{x}{2} - 2. \quad (3) 3x - \frac{9x+7}{3} = \frac{24x-4}{7} - 1.$$

$$(2) \frac{2x}{3} - \frac{x+7}{11} = \frac{x}{6} + 1. \quad (4) \frac{10x-5}{7} - \frac{15x-8}{6} = \frac{2}{5}.$$

$$(5) \frac{34x+7}{6} - \frac{2}{9}(14x-2) = \frac{22x-2}{3} - \frac{1}{2}(6x-1).$$

$$(6) \frac{x-1}{5} = \frac{3x+1}{15} - \frac{2x-8}{7x-32}.$$

$$(7) \frac{6x-2}{4x-5} + \frac{6x-21}{5} = \frac{18x-33}{15}.$$

$$(8) \frac{2x-4}{7x-13} - \frac{6x+1}{15} = \frac{1-2x}{5}.$$

$$(9) \frac{3x+3}{15} - \frac{3x-6}{5x-50} = \frac{x-9}{5}.$$

$$(10) \frac{4x+6}{6x-4} = \frac{8x+5}{12x-1}.$$

$$(11) \frac{3x-2}{3x+2} + \frac{6x+4}{9x^2-1} - \frac{3x-1}{3x+1} = 0.$$

$$12. \frac{1}{4x-3} + \frac{3}{6x-2} = \frac{6}{8x+1}. \quad 15. \frac{v+1}{v^2-26v+169} = \frac{21}{v-13}.$$

$$13. \frac{x-3}{x+3} - \frac{x+2}{x-2} = 0. \quad 16. \frac{2v-3}{v^2-7v+10} = \frac{9}{v-2}.$$

$$14. \frac{5x-2}{x+1} - \frac{x+7}{2x+1} = 3. \quad 17. \frac{1}{v-3} + \frac{1}{15} = -\frac{2}{v+17}.$$

$$(18) \frac{12}{v^2-9} - \frac{1}{v-3} = \frac{2}{v+3}.$$

$$(19) \frac{2v-1}{4v^2-10v+6} + \frac{1+2v}{2v-3} - 1 = 0.$$

$$(20). \frac{2}{v+4} - \frac{11}{v^2-16} = \frac{1}{4-v}.$$

$$21. \frac{7}{2v^2-17v+36} = \frac{2}{v-4} + \frac{1}{9-2v}.$$

$$22. \frac{1}{v-2} - \frac{1}{v-6} = \frac{1}{v-4} - \frac{1}{v-8}.$$

$$23. \frac{.5 - 1.5v}{4} = \frac{.75v + .25}{4} - .5v.$$

$$24. \frac{1.2v}{2.5} + 30v = \frac{4v+1}{.2} - 18.1.$$

$$25. 15 + \frac{v-.25}{.25} = \frac{3v+2.25}{.375}.$$

$$26. \frac{.12v}{.02} - 1.34 = \frac{.12v}{.5} + \frac{2v}{5}.$$

$$27. \frac{.6v+.34}{.75} + 3 + \frac{v-2.8}{2.5} = \frac{5v}{3}.$$

$$28. \frac{.1v-4}{.625} + 560 = \frac{.1v-2}{.5}.$$

Solve for x :

$$29. \frac{m-c}{x} + \frac{m}{x} = 1 + \frac{m-c}{x}.$$

$$33. \frac{ax+m}{ax-n} = \frac{c}{a}.$$

$$30. \frac{a}{a-x} = \frac{c}{c-x}.$$

$$34. \frac{a}{x} - \frac{c}{x} = a^2 - c^2.$$

$$31. \frac{a}{m+mx} = \frac{a}{n-nx}.$$

$$35. \frac{x+m}{x-a} = \frac{x+n}{x-c}.$$

$$32. \frac{mx^2}{ax-c} = m + \frac{mx}{a}.$$

$$36. \frac{a}{a+x} + \frac{1}{a} = \frac{a+x}{ax}.$$

$$37. \frac{x^2}{a^2-ax} = a + 1 - \frac{x}{a}.$$

$$38. \frac{mx+n}{mx+n} + \frac{n+mx}{r} = \frac{mx}{r} + \frac{s}{2r} + \frac{n}{r}.$$

$$39. \frac{2x}{m} + \frac{n-x}{m} + \frac{m-1}{n-mn} = \frac{n}{m} + \frac{x^2-n}{mx}.$$

$$40. \frac{n}{x^2-m^2} = \frac{m}{x-m} - \frac{n}{x+m}.$$

$$41. \frac{mx+c-d}{nx-a+b} = \frac{mx+c-a}{nx-d+b}.$$

Solve:

$$42. v = \frac{4\pi ab^2}{3}, \text{ for } a.$$

$$43. S = \frac{a}{1-r}, \text{ for } r.$$

$$44. \frac{x}{a} + \frac{y}{b} = 1, \text{ for } b.$$

$$45. S = \frac{n}{2}[2a + (n-1)d], \text{ for } d.$$

$$46. S = (A + B + C - 2)r^2, \text{ for } C.$$

$$47. \pi r_1^2 - \pi r_2^2 = A, \text{ for } r_1 - r_2.$$

$$48. R_t = R_0(1 + at), \text{ for } a.$$

$$53. \frac{a+b}{a-b} = \frac{c+d}{c-d}, \text{ for } a.$$

$$49. \frac{1}{f_1} + \frac{1}{f_2} = \frac{2}{R}, \text{ for } R.$$

$$54. \frac{n}{u} + \frac{1}{u} = \frac{n-1}{r}, \text{ for } n.$$

$$50. \frac{1}{r_1} - \frac{1}{r} = \frac{K}{C}, \text{ for } C.$$

$$55. \frac{n_2}{u} + \frac{n_1}{u} = \frac{n_2 - n_1}{r}, \text{ for } n_2.$$

$$51. w = \frac{n_2 - n_1}{n - 1}, \text{ for } n.$$

$$56. \frac{1}{F} = (n-1)\left(\frac{1}{r_1} + \frac{1}{r_2}\right), \text{ for } n.$$

$$52. \frac{1}{c} = \frac{1}{c_1} + \frac{1}{c_2} + \frac{1}{c_3}, \text{ for } c.$$

$$57. \frac{1}{F_1} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1 f_2}, \text{ for } F_1.$$

$$58. \frac{L}{l} + \frac{M}{n} + .001 K = \frac{L_1}{n} + M, \text{ for } L.$$

$$59. n = \frac{n_0(v+w-v_0)}{v+w-v_s}, \text{ for } v_s.$$

$$60. \text{Solve for } v_s \text{ in Exercise 59, if } w = 0.$$

$$61. a^2 = b^2 + c^2 - 2bcx, \text{ for } x.$$

Problems

1. If 375 is divided by a certain number the partial quotient is 17 and the remainder is 18. Find the number.
2. The numerator of a certain fraction is 13 more than the denominator. If 5 is added to the numerator and 6 subtracted from the denominator the value of the fraction equals 2. Find the fraction.
3. If a number is increased by 63 and the sum is divided by 17 and the quotient multiplied by 4, the result is 2 less than the number. Find the number.
4. The sum of the digits of a three-digit number is 15. The units' digit is 2 greater than the hundreds' digit and 2 less than the tens' digit. Find the number.
5. The sum of the digits of a two-digit number is 7. If 27 be added to the number the result is expressed by the digits in reverse order. Find the number.
6. If 36 is subtracted from a number of two digits the result is expressed by the same digits in reverse order. The tens' digit is twice as great as the units' digit. Find the number.
7. The numerator of a fraction is 1 less than its denominator. The sum of the fraction and its reciprocal is $2\frac{1}{42}$. Find the fraction.
8. If 1 be added to the denominator of a certain fraction the value is less than the original by $\frac{1}{15}$. The numerator is 1 greater than the denominator. Find the fraction.
9. A certain number added to its reciprocal equals 2. Find the number.
10. A certain number subtracted from its reciprocal gives a difference of 2.1. Find the number.
11. The numerator and denominator of a certain fraction are consecutive odd numbers. If 1 be added to both terms of the fraction the value of the result is .9. Find the fraction.
12. The numerator of a certain fraction is 14 greater than its denominator. If 12 be added to the numerator and 10 be subtracted from the denominator the value of the resulting fraction is double the first. Find the fraction.

13. Part of twelve hundred dollars is invested at 5% and the remainder at 6%. The annual yield is \$62. Find each investment.

14. A certain sum is invested at $5\frac{1}{2}\%$ and \$300 more at 6%, giving a total annual income of \$64. Find each investment.

15. A certain sum invested at $4\frac{1}{2}\%$ yields annually \$3 more than a sum \$200 less, invested at 6%. Find each investment.

16. Part of \$2500 is invested at 4% and the remainder at 5%. The latter yields annually \$10 less than the 4% investment. Find each investment.

17. Eighteen hundred dollars yields annually \$104. Two hundred dollars more is invested at 6% than at the other rate. Find the other rate and each investment.

18. A man has \$4000 invested, part at 5% and part at 6%. The annual income is \$216. Find each investment.

19. Six years ago A was three times as old as B. Six years from now B will be half as old as A. Find the age of each now.

20. If A were 4 years older he would be twice as old as B. Eight years ago B was three sevenths as old as A. Find the age of each now.

21. A woman's age 3 years ago was five sixths of her husband's age. The sum of their ages is 50 years. Find the age of each.

22. A father's age is five times his son's age. In 2 years the son will be one fourth as old as his father. Find the age of each now.

23. A's age is three fourths of B's. Six years from now it will be four fifths of B's. Find their ages now.

24. Eight years ago B's age was twice A's. Four years hence it will be five fourths of A's. Find the age of each now.

25. A is 40 years old and B is 13. In how many years will A be twice as old as B?

26. How many years ago was B three times as old as A if they are now 50 and 38 respectively?

27. The sum of the ages of a father and a son is 38 years. In 5 years the father will be three times as old as his son. Find the age of each now.

28. A collection of dimes and quarters amounts to \$5.90. Find how many of each there are if there are 35 coins.

29. A collection of nickels and dimes amounts to \$4.85. There are 72 coins. Find the number of each.

30. A collection of nickels, dimes, and quarters contains 120 coins and amounts to \$13.65. There are 4 more dimes than nickels. Find the number of each.

31. A can do a piece of work in 10 days, and B can do it in 6 days. Find the time required when both work together.

32. A can do a piece of work in 6 days, B can do the same work in 10 days, and C and D would consume 8 days each. Find the time required when all work together.

33. A can do a piece of work in 8 days. A and B together can do the same piece in $4\frac{4}{9}$ days. How long will B require alone?

34. A can do a piece of work in $3\frac{2}{5}$ days, B can do it in 3 days, and A, B, and C together require $1\frac{5}{18}$ days. How long will C require alone?

35. A can do a piece of work in 10 days. After he has worked 1 day B joins him and they finish the work in 3 days. Find the time which B would require alone.

36. Two runners run in the same direction around a circular 88-yard track, the first one in 18 seconds, the other one in 16 seconds. If they start together find the number of seconds before they will be together again.

37. If in Problem 36 the two runners start at the same time and go in opposite directions when will they meet and where?

38. Two motor cars starting together run in the same direction around a circular one-mile track, one at the rate of 44 feet per second and the other at the rate of 40 feet per second. Find how soon they will be together and how far they will have gone at that time.

39. If in Problem 38 the two cars start at the same time and run in opposite directions when will they meet and where?

40. The earth goes round the sun once in 365 days, and the planet Venus makes a similar movement in 225 days. When two planets are on the same side of the sun and the three are in a straight line the two planets are said to be in conjunction. Show that the interval from one conjunction to the next is about 587 days.

41. Jupiter goes round the sun in 11.86 years, and Saturn does this in 29.46 years. Find the number of years from one conjunction of these two planets to the next one.

42. If V_0 is the volume of a quantity of gas at zero degrees centigrade and V_t equals its volume at any temperature t , then $V_t = V_0 (1 + .00367 t)$. A quantity of gas at zero degrees centigrade has a volume of 250 cubic inches. Find the volume when the temperature is 50° centigrade.

43. If water from a reservoir discharges through a "smooth" outlet h feet below the surface, then the velocity of the water in feet per second is expressed by the formula $V = \sqrt{2gh}$ where $g = 32$. Find the velocity of the water flowing out of a smooth nozzle at a point 16 feet below the surface of the water.

CHAPTER XVIII

LINEAR SYSTEMS

Exercises

Solve the following systems, and check as directed by the teacher:

1. $x + y = 7,$
 $y - x = 3.$
2. $3x + 2y = 1,$
 $3x - 7y = 10.$
3. $x + 3y = -13,$
 $5x - 3y = -29.$
4. $3x + 2y = 7,$
 $10x - 5y = 0.$
5. $14p - 19q = 80,$
 $3p + 2q = 5.$
6. $9x + 2y = -2,$
 $5x - 3y = 3.$
7. $10x - 3y = 5,$
 $12x + 7y = 59.$
8. $2m - 3n = -12,$
 $6m + 5n = -8.$
9. $x + 3y = -5,$
 $5x - 2y = 9.$
10. $2x + 15y = 35,$
 $3x - 10y = -45.$

$$\begin{aligned} 11. \quad x - 12y &= -1, \\ 13x + 2y &= -13. \end{aligned}$$

$$\begin{aligned} 12. \quad 2x - 5y &= -3, \\ 4x + 7y &= 11. \end{aligned}$$

$$\begin{aligned} 13. \quad \frac{4x}{3} - \frac{12y}{5} &= 64, \\ \frac{2x}{5} + \frac{16y}{3} &= \frac{742}{15}. \end{aligned}$$

$$\begin{aligned} 14. \quad \frac{3x+y}{3} + \frac{2x+y}{2} &= \frac{7}{6}, \\ \frac{x+3y}{3} + \frac{x+2y}{2} &= -\frac{7}{6}. \end{aligned}$$

$$\begin{aligned} 18. \quad \frac{x+y-3}{4} + \frac{x-y}{3} &= -\frac{8}{3}, \\ \frac{x-y+4}{2} - \frac{x+y-1}{5} &= -\frac{1}{10}. \end{aligned}$$

$$\begin{aligned} 19. \quad \frac{x+y}{x-y} &= 0, \\ 2x + 3y &= -1. \end{aligned}$$

$$\begin{aligned} 20. \quad \frac{3x-2y+7}{3} + 2 &= 7y - \frac{4}{3}, \\ \frac{3x}{2} + \frac{7y}{6} &= \frac{13}{6}. \end{aligned}$$

$$\begin{aligned} 21. \quad \frac{3}{x} + \frac{7}{y} &= -\frac{5}{12}, \\ \frac{3}{2x} + \frac{1}{5y} &= \frac{41}{120}. \end{aligned}$$

$$\begin{aligned} 22. \quad .2x + .7y &= 5.1, \\ .3x - .01y &= 2.35. \end{aligned}$$

$$\begin{aligned} 23. \quad .7x - 1.1y &= 4, \\ 3.5x + 2.7y &= -4.6. \end{aligned}$$

$$\begin{aligned} 24. \quad 3x + .25y &= 4.25, \\ 4x + 7.5y &= 26.5. \end{aligned}$$

$$\begin{aligned} 15. \quad \frac{x+y}{5} + \frac{x-y}{3} &= \frac{26}{15}, \\ \frac{2x-y}{7} + 4y &= -\frac{131}{7}. \end{aligned}$$

$$\begin{aligned} 16. \quad \frac{9a+7b}{4} + \frac{3a}{2} &= \frac{3}{2}, \\ \frac{6a-2b}{3} - \frac{b}{2} &= -\frac{11}{2}. \end{aligned}$$

$$\begin{aligned} 17. \quad \frac{2}{x} + \frac{3}{y} &= \frac{7}{2}, \\ \frac{3}{x} + \frac{5}{y} &= \frac{11}{2}. \end{aligned}$$

$$\begin{aligned} 25. \quad \frac{1}{a} + \frac{1}{b} &= \frac{10}{21}, \\ \frac{1}{a} + \frac{3}{b} &= \frac{8}{7}. \end{aligned}$$

$$\begin{aligned} 26. \quad \frac{3}{m} + 5n &= 9.5, \\ \frac{4}{m} - 6n &= 1.8. \end{aligned}$$

$$\begin{aligned} 27. \quad 2x + 7y &= -1.45, \\ \frac{3x}{2} - .35y &= .8725. \end{aligned}$$

$$28. \frac{.7}{x} + \frac{.5}{y} = .85,$$

$$\frac{.2}{x} - \frac{.3}{y} = -.2.$$

$$29. 2x + 3y = -5a,$$

$$6x - 7y = 33a.$$

$$30. \frac{5x}{2} - \frac{3y}{7} = \frac{47m}{14},$$

$$3x - 5y = 13m.$$

$$31. .3x + .4y = 1.1a,$$

$$5.3x + 2.3y = 9.9a.$$

$$32. mx + y = 4m,$$

$$\frac{x}{3m} + \frac{2y}{5m} = \frac{5+2m}{5m}.$$

$$33. 3x + 2y = 5b,$$

$$x - 5y = 13b.$$

$$34. \frac{2}{x} + \frac{5}{y} = c,$$

$$\frac{3}{x} - \frac{5}{y} = d.$$

$$35. ax + by = c,$$

$$dx - ey = f.$$

$$36. .4x + .5y = 1.4a,$$

$$.3x - .2y = -.1a.$$

$$37. x + y = 4,$$

$$2x + 5y = 12\frac{1}{2}.$$

$$38. x + 5y = -1,$$

$$5x + 2y = 8\frac{4}{5}.$$

$$39. 2x + 3y = -3,$$

$$\frac{x}{3} - \frac{y}{2} = \frac{1}{6}.$$

$$40. \frac{x+y}{2} + \frac{3x-5y}{6} = \frac{5}{3},$$

$$\frac{x}{4} + \frac{x-y}{3} = \frac{5}{12}.$$

$$41. \frac{2x}{13} - \frac{5y}{6} = -\frac{41}{156},$$

$$\frac{x}{3} + \frac{y}{10} = \frac{9}{20}.$$

$$42. \frac{1}{x} + \frac{2}{y} = \frac{1}{6},$$

$$\frac{3}{x} + \frac{5}{y} = \frac{1}{6}.$$

Problems

1. A collection of 23 coins consists of quarters and nickels. How many coins of each denomination are there if the collection is worth \$3.15?

2. The sum of the digits of a two-digit number is 12. If 36 is added to the number the sum is equal to the original number with the digits reversed. Find the original number.

3. A is now 11 years younger than B. Last year B was two and one half times as old as A. Find the age of A and of B.

4. The sum of the reciprocals of two numbers is equal to $\frac{3}{5}$. The sum of the numbers is 12. Find the numbers.

*Reciprocal = Unity $\div$ by the no. in the
The turned upside down.*

5. Two numbers are in the ratio of 1 to 2. The reciprocal of the first plus twice the reciprocal of the second is equal to $\frac{1}{3}$. What are the two numbers?

6. An airplane flying against the wind covers a distance of 300 miles in 6 hours. The return trip took $1\frac{2}{7}$ hours. What was the speed of the airplane, and how fast was the wind blowing?

7. A can do a piece of work in 3 days less than B. Working together they take 2 days. How long does each require to do the job alone?

8. A and B together can do a job in 7 hours. A and B work together for 5 hours, and then B finishes the job in 16 hours more. How long would it take A and B each to do the whole job?

9. A sum of \$50,000 is invested, partly at 5% and the remainder at 7%. The annual income is \$3100. Find the amounts invested at each rate of interest.

10. A train starts from A and travels west at a speed of 40 miles per hour. An hour later a second train starts from a point 50 miles east of A and travels west at a speed 15 miles per hour greater than that of the first train. How long before the two trains will be together?

11. There are n coins in a collection of nickels and pennies. How many of each are required to make the collection worth p cents?

12. A fraction has a value of $\frac{a}{b}$. If c is added to both numerator and denominator, the fraction becomes equal to d . Find the original fraction.

13. A works 4 times as fast as B. The two working together can complete a job in n days. How long would it take each, working alone, to do the same work?

14. Two boys balance on a seesaw when they are 10 feet and 5 feet from the fulcrum respectively. How much does each boy weigh if the sum of their weights is 180 pounds?

15. Find the two numbers whose sum is s and whose difference is equal to $\frac{r}{4}$.

16. A boy weighing 75 pounds balances a boy weighing 60 pounds, on a seesaw. If the distance between them is 27 feet, find the distance of each boy from the fulcrum.

17. A triangle has an altitude of 2 feet and a base of a feet. How much must the altitude be increased to double the area, if the base is reduced 3 feet?

18. A boat going m miles per hour in still water travels upstream and back in a river. If the trip up took 2 hours and the trip back took 1 hour, find the distance traveled in terms of m , and the speed of the current.

Review Exercises

Solve:

$$1. 2(x + 4) + 3(x - 5) = 21.$$

$$2. \frac{.2x + 6}{5} = 13.$$

$$3. \begin{aligned} 3x - 5y &= 29, \\ 2x - 10y &= 46. \end{aligned}$$

Divide:

$$4. a^3 + a^2b - ab^2 - b^3 \text{ by } a + b.$$

$$5. 3x^4 + 23x^3y + 12x^2y^2 - 33xy^3 - 5y^4 \text{ by } x^2 + 7xy + y^2.$$

Factor:

$$6. 21x^2 - 29x - 10.$$

$$7. x^2 + 2xy + y^2 - b^2.$$

Multiply:

$$8. 3x^2 + 7xy + y^2 \text{ by } x^2 - 7xy + 3y^2.$$

$$9. x^3 - 5xy^2 + y^4 \text{ by } x^2 + 5y^2.$$

Perform the indicated operations and reduce to simplest terms:

$$10. \left(\frac{13x}{2y} + \frac{5y}{3x} + 2 \right) \div \left(\frac{18y}{4x} - \frac{3}{3y} \right).$$

$$11. \left(\frac{x-y}{15} - \frac{4y^2}{5x} \right) \div \left(\frac{x}{15} + \frac{2xy+y^2}{5x} - \frac{2y}{15} \right).$$

Write as fractions and simplify results :

$$12. x + 5 - \frac{6}{x-2} + \frac{3}{x-1}.$$

$$13. \left(2m + 3n - \frac{2}{m-n} \right) - \left(m - n + \frac{m+n}{2} \right).$$

14. The sum of two numbers is 117 and their difference is 87. Find the numbers.

15. A certain point has an elevation above sea level of -25 feet. What is the elevation of a point twice as far above sea level? What is the elevation of a point 50 feet higher? 75 feet lower?

16. A man is 25 years older than his son. The sum of their ages 5 years from now will be 65 years. What are their present ages?

17. One number is four times another number. Their sum is 20. Find the numbers.

18. Three boys have 30 marbles divided between them in the proportion of 1, 2, and 3. How many marbles does each boy have?

19. If New York State has 2 million less than twice as many inhabitants as New York City, and New York State outside of the city has two thirds as many inhabitants as New York City, find the size of New York City.

20. A local train starts out on a run of 150 miles. Two hours later an express train, traveling 20 miles per hour faster than the local, starts out on the same run. If the two trains reach their destination at the same time, find the speed of each train.

21. The numerator of a certain fraction is 2 less than the denominator. If 5 is added to both numerator and denominator, the resulting fraction is equal to the original fraction with 1 added to the numerator. Find the original fraction.

22. The numerator of a certain fraction is 3 less than the denominator. If 5 is added to the numerator the value of the resulting fraction is $1\frac{1}{4}$. What is the value of the original fraction?

CHAPTER XIX

GRAPHS OF EQUATIONS

Exercises

Plot the following points:

1. $(2, 3)$; $(-5, 2)$; $(0, -3)$; $(10, -2)$; $(1, 1)$.
2. $(-7, -3)$; $(-2, -5)$; $(12, \frac{1}{2})$; $(1, 1.3)$; $(.4, -.5)$.

Plot the graphs of the following equations:

- | | |
|--------------------|----------------------|
| 3. $4x + 7y = 5$. | 6. $x + y = 3$. |
| 4. $2x - 3y = 0$. | 7. $-3x + 2y = 10$. |
| 5. $6x + 2 = y$. | 8. $-y - 3x = 1$. |

Solve graphically the following systems:

- | | |
|---|---|
| 9. $2x + y = 7$,
$3x - 5y = 4$. | 12. $x + 2y = -6$,
$2x - 5y = 15$. |
| 10. $x - 13y = 27$,
$3x - 11y = 25$. | 13. $2x + 2y = 4$,
$3x + 5y = 8$. |
| 11. $7x + 5y = -13$,
$3x + y = -9$. | 14. $19x + 2y = -103$,
$13x + 25y = -165$. |

Problems

1. The average force with which a falling body strikes the ground is expressed by the formula $F = \frac{Ws}{d} + W$, where F is the average force of the blow in pounds, W is the weight of the falling body in pounds, s is the distance in feet through which the body has fallen, and d is the *penetration* or the distance in feet the body sinks into the ground when it strikes. Plot this formula for F and d , when the weight, W , is 100 pounds, and the distance, s , is 50 feet. How great a force will be exerted if the body penetrates 1 foot into the ground? 3 feet? 6 inches? no distance at all? If the greatest force which the ground can supply when the body strikes is 1000

pounds, what will be the penetration of the body? What will happen if the place where the body falls cannot exert a force greater than 150 pounds?

2. A weight attached to a string and swung in a circle exerts a pull on the string expressed as follows: $F = \frac{WRn^2}{2933}$, where

F is the pull on the string in pounds, W is the weight of the body attached to the string in pounds, R is the length of the string in feet, and n is the number of revolutions per minute that the weight is swinging through. Plot this formula for a 2-pound weight attached to a 1-foot string. What is the pull on the string if the body is swung 200 revolutions per minute? 100 revolutions per minute? How fast must the weight be swung to exert a pull of 10 pounds on the string? 25 pounds? 20 pounds?

3. The area, A , of the surface of a sphere is given in the formula $A = 4\pi r^2$, where r is the radius of the sphere. Plot this formula and from the graph determine the area of a sphere of radius 5 feet, 10 feet, 2 feet. What is the radius of a sphere of area 1000 square feet? 500 square feet? 100 square feet? .

Review Exercises

Perform the indicated operations:

$$1. (2x^2 + 3x + 9x^2 - 2x^2) \div \frac{x}{3x-1}.$$

$$2. -[2x + 3y - (x - 4y)] \div \left(\frac{x^2 + 4xy - 21y^2}{3} \right).$$

$$3. \frac{2x+3y}{4} \left[\frac{3x+2y-(x-y)}{3} \cdot \frac{9x+3y}{(10x-7y)+3y-(7x-3y)} \right].$$

Solve:

$$4. 3x + 2 = 7x - 5.$$

$$6. \frac{2x}{3} - 9 = -7.$$

$$5. 11x + 3 = 5 - 12x.$$

$$7. 14y + \frac{3y}{4} = 59.$$

Factor :

$$8. 4x^2 + 4xy + y^2 - 49. \quad 9. 3ax + 2ay + 3bx + 2by.$$

$$10. 6x^2 - 11xy - 10y^2.$$

Solve the following linear systems :

$$11. \frac{2}{x} + \frac{1}{y} = \frac{5}{2},$$

$$\frac{3}{x} + \frac{2}{y} = \frac{7}{2}.$$

$$12. \frac{x+y}{3} + \frac{7x}{5} = \frac{104}{15},$$

$$\frac{2x+3y}{4} - \frac{2y}{7} = 2.$$

$$13. 3x + 10y = 32,$$

$$4x - 30y = 21.$$

$$14. \frac{3x}{2} + .7y = .54,$$

$$5x + .2y = 2.44.$$

15. Two boys caught 12 fish. If one caught 4 more than the other, how many did each catch?

16. A rectangle has a perimeter of 18 inches. If it is 1 inch longer than it is wide, find the dimensions.

17. Find three consecutive even numbers whose sum is 24.

18. A starts from a certain spot and walks north at a speed of 3 miles per hour. B starts from the same place 2 hours later and goes south on his bicycle at a speed of 10 miles per hour. How long before they will be 30 miles apart?

19. The sum of a certain number and three times its square is equal to 2. Find the number.

20. Two weights are in the ratio 2:3. Their difference is 5 pounds. Find the weights.

CHAPTER XX

SQUARE ROOT

Exercises

Find the square root of :

$$1. 25a^2 + 60ab + 36b^2.$$

$$3. 4d^2 - 20df + 25f^2.$$

$$2. 9x^4 + 42x^2y^3 + 49y^6.$$

$$4. 25n^4 + 30n^3 + 9n^2.$$

$$5. x^2 + y^2 + z^2 + 2xy - 2xz - 2yz.$$

$$6. 9x^2 - 5xy + \frac{25y^2}{36}.$$

$$7. \frac{4x^4}{25} + \frac{3x^2y^3}{5} + \frac{9y^6}{16}.$$

Find the first three terms in the approximate square root of :

$$8. 36x^2 + 5xy + 15y^2 + 25y^3.$$

$$9. 16x^2 - 15xy + 25y^2.$$

$$10. 9x^4 + 3x^3 + 16x^2 + 5x - 8.$$

$$11. x^6 + 3x^5y + 2x^4y^2 - 10x^3y^3 + 6x^2y^4 + 25xy^5 - 16y^6.$$

Find the positive square root of the following numbers to four significant figures:

$$12. 9.0653.$$

$$16. 1492.$$

$$20. 151.27389.$$

$$13. 1000.$$

$$17. 4,896,521.$$

$$21. 1.29684.$$

$$14. 1898.25.$$

$$18. 49.87652.$$

$$22. 1\frac{1}{2}\frac{8}{5}.$$

$$15. 6,276,666.$$

$$19. .00087315.$$

$$23. 5\frac{9}{11}.$$

In the following right triangles find the missing side when :

$$24. \text{Hypotenuse} = 164 \text{ inches; side} = 100 \text{ inches.}$$

$$25. \text{Side} = 25 \text{ feet; side} = 75 \text{ feet.}$$

$$26. \text{Side} = 1 \text{ mile; hypotenuse} = 7480 \text{ feet.}$$

27. Find the length of a path running diagonally across a rectangular park 100 feet wide and 150 feet long.

28. A cube has an edge of 3 inches. Find the length of the diagonal of the cube.

HINT. The diagonal of the cube is the hypotenuse of a right triangle whose sides are the edge of the cube and the diagonal of one side, respectively.

29. The area of the surface of a sphere is expressed by the formula $A = 4\pi r^2$, where r is the radius of the sphere. What is the radius of a sphere whose area is 200 square feet? 1 square mile? 1 square inch?

30. A man walks around a city block which is in the form of a square. If he could have walked along the diagonal and back, he would have saved 400 feet. Find the dimensions of the block.

31. Find the altitude and the side of an equilateral triangle whose area is 140 square feet.

Review Exercises

Simplify :

1. $[2a - 5b + 3a + 2(a - 5b) + 3a]$
 $\times \left[3a + 2b - (4a - b) + 2a \left(1 + \frac{2b}{a} \right) \right].$
2. $\frac{2a - 5b}{9a} + \frac{3a + 5b - (2a + b)}{3(a - 2b)} - \frac{16a + 2(a + b)}{9(a - 2b)}.$
3. $\frac{2a^2 + a - 6}{(a - 1)} \div \frac{(a + 2)}{2a^2 - 5a + 3} \cdot \frac{a^2 + a - 2}{(2a - 3)}.$

Factor :

4. $2ab + 8a + 12 + 3b.$
5. $10ax + 15bx + 10x + 14ay + 21by + 14y + 2az$
 $+ 3bz + 2z.$
6. $3a^3 + 8a^2 + 4a.$
7. $18x^2 + 31xy - 35y^2.$

Solve :

8. $3x + 7 = \frac{5}{2}x.$
9. $x^2 + 3x = 10.$
10. $6x^2 + 37x = 60.$
11. $2x + 3y = -2,$
 $9x + 8y = 13.$
12. $\frac{5}{x} + \frac{2}{y} = \frac{3}{2},$
 $\frac{9}{x} + \frac{3}{y} = 3.$
13. $\frac{x + 3y}{5} + \frac{2x - y}{3} = -\frac{4}{15},$
 $\frac{9x + 13y}{2} - \frac{y}{8} = -\frac{51}{8}.$

Solve graphically :

14. $6x + 4y = 7,$
 $2x = y.$
15. $\frac{x + y}{2} + \frac{y}{3} = 2, \frac{2x - 5y}{6} = \frac{7x}{5} - \frac{43}{30}.$

16. The front wheel of a wagon has a diameter 1 foot less than the diameter of the rear wheel. How far has the wagon gone when the front wheel has made 150 revolutions more than the back wheel, and the two wheels together have made a total of 750 revolutions?

17. There were 100 children at a picnic. If there had been 5 more girls there would have been half as many girls as boys. Find the number of girls and boys present.

18. A man wishes to run an aerial for a radio from a point on his house, 25 feet above the ground, to the top of a 20-foot pole at a distance of 50 feet from the house. How much wire will he need to reach between these two points?

19. A man receives as pay \$40 per week. During illness he receives half pay. For a 25-week period he received \$900. How many weeks did he work and how long was he sick?

20. It is found that a 50-foot rope stretched from the top of a flagpole reaches the ground 15 feet from the base of the pole. Find the height of the flagpole.

CHAPTER XXI

RADICALS

Exercises

Simplify:

1. $\sqrt{45}$.

7. $\sqrt{9a^3}$.

13. $9\sqrt[3]{16m^6}$.

2. $\sqrt{200}$.

8. $\sqrt{3x^4}$.

14. $\sqrt{3x(a-b)^2}$.

3. $\sqrt{108}$.

9. $4\sqrt{25y}$.

15. $3\sqrt{25z(x+y)^2}$.

4. $3\sqrt{18}$.

10. $\sqrt{x^5}$.

16. $\sqrt{(x+y)(x^2-y^2)}$.

5. $\sqrt[3]{375}$.

11. $\sqrt[3]{29x^5}$.

17. $\sqrt{3x^2+6xy+3y^2}$.

6. $2\sqrt[3]{72}$.

12. $\sqrt[4]{x^5}$.

18. $\sqrt{4a^2-16ab+16b^2}$.

Simplify and collect:

19. $\sqrt{27} + \sqrt{75}$.

23. $y^2\sqrt{\frac{x}{y}} + \frac{y}{x}\sqrt{x^3} + \frac{1}{x}\sqrt{x^3y^2}$.

20. $2\sqrt{50} + 2\sqrt{8}$.

21. $12\sqrt{32} - 7\sqrt{2}$.

24. $12\sqrt{a} + \frac{1}{2}\sqrt{4a} + \frac{1}{a}\sqrt{a^3} + a\sqrt{\frac{1}{a}}$.

22. $\sqrt{x^3} + x^2\sqrt{\frac{1}{x}} + x\sqrt{x}$.

25. $\sqrt{x^3} + 12x\sqrt{x} + \sqrt{81x^7}$.

Simplify:

26. $\sqrt{\frac{5}{12}}$.

32. $2\sqrt{\frac{1}{2}} + 5\sqrt{\frac{3}{2}\frac{2}{5}}$.

27. $\sqrt{\frac{9}{10}}$.

33. $3\sqrt{27} + 8\sqrt{75} - 2\sqrt{12}$.

28. $3\sqrt{\frac{18}{24}}$.

34. $\frac{8\sqrt{125} + 20\sqrt{\frac{1}{5}} - 10\sqrt{45}}{3\sqrt{80}}$.

29. $5\sqrt{\frac{21}{15}}$.

35. $3\sqrt{44} + \sqrt{\frac{4}{11}} + 3\sqrt{\frac{81}{44}}$.

30. $12\sqrt{\frac{2}{3}}$.

36. $45\sqrt{\frac{1}{50}} + 5\sqrt{162} - 18\sqrt{\frac{3}{4}} + 14\sqrt{\frac{1}{18}}$.

31. $2\sqrt{\frac{x^2y}{3}}$.

37. $15\sqrt{(16a^2 + 16a + 4)a} - 5\sqrt{(2a + 1)^2a^5}$.

Perform the indicated operation and simplify results:

38. $3\sqrt{32} \cdot 10\sqrt{8}$.

45. $\sqrt{3}(\sqrt{3} - \sqrt{2})$.

39. $5\sqrt{75} \cdot 6\sqrt{147}$.

46. $\sqrt{10}(\sqrt{2} - \sqrt{5})$.

40. $a\sqrt{8} \cdot b\sqrt{50}$.

47. $(\sqrt{3} - \sqrt{10} + 2\sqrt{5})(-2\sqrt{2})$.

41. $2\sqrt{3} \cdot \sqrt{5}$.

48. $(\sqrt{2} + 3)(3\sqrt{2} - 5)$.

42. $\sqrt{2} \cdot \sqrt{10}$.

49. $(2\sqrt{3} - 5\sqrt{2})(9\sqrt{2} + 3\sqrt{3})$.

43. $\sqrt{\frac{9x}{13}} \cdot \sqrt{\frac{13x}{9}}$.

50. $(4 + 3\sqrt{7} + 2\sqrt{10})(5\sqrt{2} - \frac{1}{2}\sqrt{3})$.

51. $(2\sqrt{x} + 5\sqrt{y})(3\sqrt{x} - 6\sqrt{y})$.

44. $5\sqrt{\frac{2}{3a}} \cdot 2\sqrt{\frac{5a}{3}}$.

52. $\sqrt{(x+y)(x-y)} \cdot \sqrt{3x-3y}$.

53. $3\sqrt{(x^2-x-6)} \cdot \sqrt{(x^2+5x+6)}$.

54. $(\sqrt{x+y} + 3\sqrt{y} - \sqrt{y-x})(2\sqrt{y})$.

Find the radical expression having a coefficient 1, which is equivalent to each of the following:

55. $2\sqrt{2}$.

56. $5\sqrt{10}$.

57. $15\sqrt{3}$.

58. $2\sqrt{75}$.

Rationalize the denominator of the following expressions:

59. $\frac{\sqrt{30}}{\sqrt{5}}$.

61. $\frac{\sqrt{2} - \sqrt{8}}{\sqrt{2}}$.

63. $\frac{2}{\sqrt{x}}$.

60. $\frac{\sqrt{2}}{\sqrt{6}}$.

62. $\frac{\sqrt{6} + \sqrt{15}}{\sqrt{3}}$.

64. $\frac{15\sqrt{a}}{4\sqrt{3}b}$.

65. $\frac{5\sqrt{xy}}{2x\sqrt{y}}$.

66. $\frac{4\sqrt{75}}{6\sqrt{5\frac{1}{3}}}$.

67. $\frac{3\sqrt{x}}{4\sqrt{y}}$.

68. $\frac{9\sqrt{xy}}{10\sqrt{zx}}$.

Find to three significant figures the value of:

69. $9 + 6\sqrt{2}$.

71. $\frac{3}{\sqrt{5}}$.

73. $\frac{18\sqrt{2}}{6\sqrt{3} - 2\sqrt{5}}$.

70. $2\sqrt{3} + 8\sqrt{2}$.

72. $\frac{\sqrt{91} + 3\sqrt{6}}{\sqrt{2}}$.

74. $\frac{3\sqrt{2}}{7\sqrt{10}}$.

Write in radical form and simplify results:

75. $9x^{\frac{1}{3}}y^{\frac{9}{10}}$.

77. $5(xy)^{\frac{7}{5}}$.

79. $7(a^{\frac{2}{3}})^3(b^{\frac{7}{2}})^{\frac{2}{3}}$.

76. $2a^{\frac{5}{2}}b^{\frac{13}{10}}$.

78. $5x^{\frac{12}{5}}y^{\frac{21}{10}}$.

80. $3(x^{\frac{1}{2}})^{\frac{5}{2}} \cdot 4(y^{\frac{2}{3}})^{\frac{3}{5}}$.

Problems

1. One side of an equilateral triangle is 24 feet. Find the altitude and the area.

2. A man walks 2 miles east and then $\frac{1}{2}$ mile north. How far is he from his starting point?

3. The area of an isosceles right triangle is $12\frac{1}{2}$ square feet. Find the length of the hypotenuse.

4. The formula for the area of a sphere is $A = 4\pi r^2$. Find the radius in terms of the area. Using this result determine the radius of a sphere having an area of 10 square yards.

5. How many spheres of radius 1 foot will have an area of 100 square feet?

6. If an airplane can climb at an angle of 30° with the horizontal, how far will it rise in 15 minutes when its speed with respect to the horizontal is 50 miles per hour?

7. The area of a square is 5 square feet. Find the length of the diagonal.

8. A tent is made from a sheet of canvas 13 feet wide stretched over a ridgepole. The edges of the tent are pegged to the ground. How high should the ridgepole be, if the tent is to have a spread of 8 feet on the ground?

9. A 20-foot pole leans against a point on a wall 15 feet from the ground. How far from the foot of the wall is the lower end of the pole?

Review Exercises

If $a = 5$, $b = 2$, $m = -3$, $x = 1$, and $y = 4$, evaluate the following expressions, carrying all answers to three significant figures:

$$1. 2\sqrt{bx} + m\sqrt{y}.$$

$$5. 3m\sqrt{x^2 + 3xy - 9}.$$

$$2. a\sqrt{b^3x^2} - x\sqrt{a} + \frac{y}{m}.$$

$$6. \frac{3a\sqrt{m^2 + 2mx + y^2}}{x + \sqrt{xy} + y}.$$

$$3. \frac{m + x\sqrt{a}}{\sqrt{x} + y} + \frac{a\sqrt{y} + 2}{m}.$$

$$7. \frac{3\sqrt{-m^3}}{2\sqrt{3}y} + 2\frac{\sqrt{x}}{\sqrt{b}}.$$

$$4. \sqrt{\frac{m}{-x}} + 3\sqrt{2a + m}.$$

$$8. \frac{19\sqrt{\frac{x}{m} + \frac{1}{b^2} + a}}{2a - 15m + (y - a^2)}.$$

Solve for x and y :

$$9. \frac{2x + 3}{9} = 1.$$

$$10. \frac{x + m}{x + n} = 2.$$

$$11. \frac{x + 2a}{3a} = \frac{x - 5a}{a}.$$

$$12. \frac{3x + 5y}{15} = \frac{2}{3} + y,$$

$$14. \frac{2}{5x} + \frac{3}{y} = -\frac{13}{5},$$

$$2x + 9y = 13.$$

$$\frac{9}{3x} + \frac{7}{y} = -4.$$

$$13. \begin{aligned} 2x + 5y &= -4, \\ 3x - 21y &= 201. \end{aligned}$$

Extract the positive square root of :

15. $81x^2 + 234xy + 169y^2$.

16. $4x^2 + 12x + 9$.

17. $9x^2 + 42xy + 12xz + 28yz + 49y^2 + 4z^2$.

18. $a^2 + 6ab - 10ac - 30bc + 9b^2 + 25c^2$.

19. A cloud is seen by one observer to be directly overhead. A second observer sights the same cloud along a line making an angle of 60° with the horizontal. If the distance between observers is 2 miles, find the height of the cloud.

20. The sum of three consecutive odd numbers is 69. Find the numbers.

21. A man is now three times as old as his son. Five years ago the man was five times as old as his son. Find their present ages.

22. A has \$20 now while B has only \$5. A increases his money at the rate of \$10 per week, while B increases his money at the rate of \$13 per week. How long before B will have as much money as A?

CHAPTER XXII

QUADRATIC EQUATIONS

Exercises

(Obtain all answers correct to three significant figures.)

Solve by completing the square and check as directed by the teacher :

1. $9x^2 + 24x + 16 = 0$.

2. $6x^2 + x = 2$.

3. $10x^2 + 17x - 20 = 0$.

4. $14x^2 - 5x = 39$.

5. $7x^2 + 6 = 23x$.

6. $6x^2 - x = 57$.

7. $22y^2 - 49y - 15 = 0$.

8. $6x^2 - x - 1 = 0$.

9. $9x^2 + 18x - 2 = 0$.

10. $x^2 + 3x - \frac{2}{3} = 0$.

11. $49x^2 - 41x - 28 = 0$.

12. $2x^2 - 5x + 3 = 0$.

13. $x^2 + 9.8x - 5.15 = 0$.

14. $x^2 + .95x + .15 = 0$.

15. $x + \frac{2}{3} = \frac{3}{x}$.

16. $\frac{9}{x+3} = x+2.$

20. $2x^2 - 13x - 6 = 0.$

21. $5x^2 - 10x + 2 = 0.$

17. $\frac{2x+3}{x-3} - \frac{x+2}{x-2} - 8 = 0.$

22. $12x^2 + 3x - 8 = 0.$

23. $13x^2 + 25x - 2 = 0.$

18. $x^2 + 2x - 9 = 0.$

24. $x^2 + 51x + 1 = 0.$

19. $13x - x^2 + 21 = 0.$

25. $3x^2 - 4x + 1 = 0.$

Solve the following systems, and check as directed by the teacher:

26. $3x^2 + 2xy + 5y = 3,$
 $3x + 2y = 4.$

28. $14x^2 - 9xy + y^2 = 6,$
 $12y - 41x = -29.$

27. $x^2 + xy + y^2 = 3,$
 $9x - 5y = 4.$

29. $7x^2 + 3xy - 10y^2 = 7,$
 $2x + 15y = 2.$

Problems

(Reject all results which do not satisfy the conditions of the problem.)

- Find two consecutive numbers whose product is 132.
- Find two consecutive odd numbers whose product is 399.
- What three consecutive even numbers have a sum equal to one half the product of the two larger numbers?
- A man takes an express train from A to B, a distance of 100 miles. He then returns on a local train. The total time required was $5\frac{1}{3}$ hours. If the express train traveled 20 miles per hour faster than the local train, find the speed of each.
- A rectangle has a diagonal of 15 inches. If the length is 3 inches more than the width, find the area of the rectangle.
- A man drops a stone from an airplane and sees it strike the water 5 seconds later. Find the height of the airplane and the vertical speed of the stone when it struck the water.
- A man on the top of a 200-foot cliff drops a stone over the edge into the water. How fast was the stone traveling when it reached the water?
- The sum of the squares of two numbers is 65. Their difference is 3. Find the numbers.

9. A certain sum of money yields \$600 annual interest. If the principal were reduced \$5000 and the interest rate raised 1%, the annual income would be reduced \$100. Find the original principal and rate of interest.

10. The altitude of an isosceles triangle is 8 inches and its perimeter is 32 inches. Find the sides of the triangle.

11. The perimeter of a rectangle and its area are both expressed by the number 28.8. Find the dimensions of the rectangle.

Review Exercises

Divide:

1. $3x^3 + 9x^2 + 7x^2y - 33xy^2 - 4y^2 + 18y^3$ by $3x - 2y$.
2. $34 - 45ab - 81b^2 - 9a^2b^2 - 21ab^3 - 10b^4$ by $2 - 3ab - 5b^2$.
3. $21x^2 - 22ax^2 + 41bx - 4cx - c^2 - 8a^2x^2 - 32abx - 6acx + 18b^2 - 7bc$ by $3x - 4ax + 2b - c$.

Factor:

4. $ad + ae + af + bd + be + bf + cd + ce + cf$.
5. $6x^2 + 4x - 2$.
6. $2a^2 + 6ab - 3ac - 9bc$.
7. $2x^2 + xy - 15y^2$.

Find the positive square root of:

8. $9x^2 + 12xy + 54x + 36y + 4y^2 + 81$.
9. $4a^2 + 12ab - 4ac + 9b^2 + c^2 - 6bc$.
10. 4,893,275 to four significant figures.
11. .0132896 to the nearest hundredth.
12. $2\frac{1}{3}\frac{5}{2}$ to the nearest thousandth.

Solve for the unknown involved:

- | | |
|------------------------------|--|
| 13. $2x = 7 + x$. | 19. $\frac{2x}{3} + \frac{y}{2} = \frac{1}{2}$, |
| 14. $12x^2 = 36x$. | $\frac{9x}{4} - \frac{3y}{5} = -\frac{3}{5}$. |
| 15. $5x^3 + 10x = 0$. | 20. $.25x - .5y = -.275$, |
| 16. $24x^2 - 22x - 21 = 0$. | $.37x + .13y = -.3635$. |
| 17. $25x^2 - 30x + 2 = 0$. | |
| 18. $2x + 3y = 1$, | |
| $9x - 2y = 20$. | |

21. A man has a tent made in the form of a rectangular sheet of canvas 10 feet by 15 feet. The tent is set up by stretching the canvas over a rope and pegging the two 10-foot sides to the ground. How high above the ground must the ridge be to give the maximum number of cubic feet under the tent?

HINT. Graph the formula for the volume of the tent in terms of the altitude.

22. A rectangular figure has a perimeter of 100 feet. Find the maximum area of such a figure.

23. The diagonal of a rectangle is 75 feet. Find the length and breadth, if the perimeter is 200 feet.

24. A freight train has 20 cars. There are one more than three times as many box cars as tank cars, and 4 more flat cars than tank cars. How many cars of each kind are there?

25. Two men start at points 2 miles apart and walk toward each other. If one man goes 3 miles per hour and the other man walks $3\frac{1}{2}$ miles per hour, find how long before they meet.

SUPPLEMENTARY CHAPTER

INTRODUCTION TO TRIGONOMETRY

Exercises

Solve the following right triangles, obtaining all results to three significant figures:

1. $A = 13^\circ 10'$, $b = 27$ feet.
2. $B = 87^\circ 9'$, $c = 100$ feet.
3. $B = 40^\circ 50'$, $a = 13\frac{1}{2}$ feet.
4. $a = 32$ feet, $c = 75$ feet.
5. $A = 19^\circ 32'$, $b = 2$ inches.
6. $b = 13$ yards, $a = 29$ yards.
7. $A = 15^\circ 37'$, $c = 101$ inches.
8. $B = 48^\circ 2'$, $c = \frac{13}{25}$ rod.

9. $A = 22^\circ 20'$, $b = 1$ mile.
10. $B = 5^\circ 39'$, $a = 2.5$ feet.
11. $B = 17^\circ 39'$, $b = 23.7$ inches.
12. $A = 52^\circ 18'$, $a = 15.7$ feet.

In the following triangles one side and an adjacent angle are given. Find an altitude of the triangle:

- | | |
|--|------------------------------|
| 13. 25 yards, $23^\circ 45'$. | 16. 1 inch, $37^\circ 11'$. |
| 14. 16 feet, $72^\circ 12'$. | 17. 32 feet, $12^\circ 1'$. |
| 15. $2\frac{1}{2}$ miles, $78^\circ 23'$. | 18. 17 feet, $32^\circ 5'$. |

Problems

1. A telegraph pole 30 feet high casts a shadow 25 feet long. Find the angle of elevation of the sun.

2. A stake driven into the ground at an angle of 75° with the horizontal casts a shadow 3 feet long when the sun is directly overhead. Find the length of the stake.

3. A surveyor finds that if three points, A , B , and C , are connected by straight lines, the angles formed at the points B and C are $32^\circ 19'$ and 90° respectively. If the distance from B to C is 423 feet, find the distance of A from both B and C .

4. The angle of repose for a certain sort of loose earth is 40° . What will be the area of cross section of a pile 10 feet high? What is the width of the base of the pile?

5. A boat travels northeast 2 miles and then turns and travels northwest for 5 miles. How many miles is it north of its starting point? east or west of its starting point?

6. A sailing vessel has a mast 75 feet high. A stay running from the masthead to the rail is 80 feet long. Find the width of the boat at the point where the mast is placed.

7. A Maypole 20 feet high is fitted with ribbons 30 feet long attached to the top of the pole. Find the radius of the greatest circle in which the children may dance around the pole, if the lower ends of the ribbons are held 3 feet above the ground.

8. A man leaves a 10-foot ladder leaning against a house so that the ladder makes an angle of $78^{\circ} 15'$ with the horizontal. How far up on the side of the house can he paint, if he can reach 6 feet above the end of the ladder?

9. A man places two ladders against a house. The foot of each ladder is 5 feet from the side of the house. If one ladder is 10 feet long and the other is 15 feet long, find how far the top of the longer ladder is above the top of the shorter ladder.

SAMPLE NEW-TYPE EXAMINATIONS¹

PREPARED AT THE REQUEST OF THE COLLEGE ENTRANCE EXAMINATION BOARD FOR THE INFORMATION OF TEACHERS AND TO FACILITATE EXPERIMENTS AND OBSERVATIONS IN SECONDARY SCHOOLS WITH A VIEW TO DETERMINING THE VALUE OF SUCH TESTS, BY THE INSTITUTE OF EDUCATIONAL RESEARCH, TEACHERS COLLEGE, COLUMBIA UNIVERSITY

INSTRUCTIONS

The time allowed for each of these tests is 90 minutes. When the signal to go is given, begin with No. 1 and do as many as you can before time is called. In general, do No. 1 first, then No. 2, then No. 3, and so on until all are done. However, if you do not know how to do some of them, go ahead to the next ones. That is, first do all that you are sure you can do; then go back and do the others if you have time. Write your answers very plainly. If you finish before the time is called, go back and perfect your work. Do not hand in your paper until time is called.

¹ Reprinted through the courtesy of the Institute of Educational Research, Teachers College, Columbia University, and the College Entrance Examination Board.

TEST I

Perform the indicated operations:

1. $(5a^2 + 4 - 3a^2) + (a^2 - 2 - 7a^2 - 5)$.
2. $(-2a^3 - 10a - 4a^3) + (5a + 3a^3 - 4a)$.
3. From $3a + 4b$ subtract $5a - 9b - 3c$.
4. From $5a - b - 2c$ subtract $3c - 3a$.
5. $8a + 8b - (3a + 6b)$.
6. $(5d - e) - (7e + 2f)$.
7. $7d \times 2de^2$.
8. $de^2 \times d^2e$.
9. $8 - 5(d + 2)$.
10. $4e^2 + e(-4e - 3)$.
11. $5np - 3p(4n + 3p)$.
12. $m + \frac{8m^2n^2}{mn}$.
13. $4m + \frac{2m^2np^3}{mp}$.
14. $\frac{m^2np}{mn} - \frac{m^2n - mp}{m}$.
15. $(m^5n)(m^2n^3)$.
16. $(2a - 7)^2$.
17. If $a = 2$, and $b = 3$, what does $5a^2 - 2ab$ equal?
18. If $a = .7$, and $b = 1.2$, what does $2a^2 - 5ab$ equal?
19. If $a = 1$, $b = 2$, $c = .4$, and $d = 100$, what does $a^2 - b + cd$ equal?
20. If $a = 12$, $b = 6$, $c = 5$, $d = 3$, and $e = 1$, what does $\frac{2}{7}[ab + c(d - e)]$ equal?
21. If $d = 2$, $e = 3$, $f = 4$, what does $\frac{df}{d+e}$ equal?
22. $d = \frac{ef}{g}$. What does f equal?
23. $\frac{e}{W} = \frac{r}{R}$. What does W equal?
24. $\frac{PV}{T} = \frac{P_1V_1}{T_1}$. What does V equal?
25. $4q = 7q + 5$. What does q equal?
26. $15 = 7w - 4$. What does w equal?
27. $\frac{V}{4} = a - 2$. What does V equal?
28. $\frac{6}{2-u} - \frac{11}{3-u} = 0$. What does u equal?
29. $P = \frac{b_1b_2}{c_1 + c_2}$. b_1, b_2, c_1 , and c_2 are all positive.

Write "larger" or "smaller" or "you cannot know" on the dotted lines.

If b_1 becomes larger, P will become

If b_2 becomes larger, P will become

If c_1 becomes larger, P will become

If c_2 becomes larger, P will become

$$30. N = \frac{ab}{c} - d(e-f) + \frac{1.414}{g + \frac{1}{h}}. \quad a, b, c, d, e, f, g, \text{ and } h \text{ are all positive.}$$

Write "larger" or "smaller" or "you cannot know" on the dotted lines.

If a becomes larger, N will become

If b becomes larger, N will become

If c becomes larger, N will become

If d becomes larger, N will become

If e becomes larger, N will become

If f becomes larger, N will become

If g becomes larger, N will become

If h becomes larger, N will become

31. If $a = 1$, $b = 10$, and $c = 100$, express 216 in numbers and letters. Express 2.16 in numbers and letters.

32. To two times a certain number 2 is added. From three times the number 7 is subtracted. The two results are equal. What is the number?

Use cross-section paper for Nos. 33, 34, 35.

33. Make a little cross ($\times$) at the point for which $x = 2$ and $y = 3$. Make a little circle ($\circ$) at the point for which $x = -4$ and $y = 1$. Make a little triangle ($\triangle$) at the point for which $x = 1\frac{1}{2}$ and $y = -4$.

34. Draw a graph of $y = 1\frac{1}{3}x - 5$. Mark it No. 34.

35. Draw a graph of $y = \frac{1}{x} + 8$. Mark it No. 35. Draw enough of it to show clearly that you understand it.

36. Find two numbers such that:

Twice the first, if added to three times the second, equals 2.

Six times the second, if added to ten times the first, equals 7.

37. What is the length of the diagonal of a rectangle, if the sides of the rectangle are 8 ft. and 6 ft.?

38. Multiply $\sqrt{\frac{a}{b}}$ by $\frac{b}{\sqrt{a}}$.

39. Write a plus sign (+) on the dotted line if the statement is true.
Write a minus sign (-) on the dotted line if the statement is false.

$$\sqrt{\frac{ab}{cd}} = \frac{\sqrt{abcd}}{cd} \quad \text{-----}$$

$$\sqrt{a}\sqrt{b}\sqrt{c} = \sqrt{a+b+c} \quad \text{-----}$$

$$a^{\frac{1}{2}} = \frac{1}{a} \quad \text{-----}$$

$$a^{\frac{1}{3}} \times a^{\frac{1}{3}} = a^{\frac{2}{3}} \quad \text{-----}$$

$$\sqrt{\frac{1}{5}} = \frac{\sqrt{5}}{5} \quad \text{-----}$$

40. Write words and letters on the dotted line to make the statement true.

If ----- $a + b + c$ will equal abc -----

TEST II

(The instructions for this test are the same as those for TEST I.)

Perform the indicated operations:

1. $(6b^2 + 9 - 2b^2) + (4b^2 - 7 - 6b^2 - 6)$.

2. $(-6ad - 2c + 2ad) + (7c - 3ad + 4c)$.

3. From $8a - b - 4c$ subtract $5b - 8c$.

4. From $4a - 5b$ subtract $2a - 9b + 5c$.

5. $a - 7b - (b - 3c)$.

6. $7de^2 - d(2de - 3)$.

7. $6d \times 4de^2$.

8. $5de^2 \div 5def$.

9. $8 - 2(d + 3)$.

10. $5d^2 \div d(-3d - 3)$.

11. $2np + p(3m - n)$.

12. $m + \frac{5m^2np}{mn}$.

13. $m + \frac{m^3n^2p^2}{mnp}$.

14. $m - \frac{6m - 9n}{3}$.

15. $(\frac{1}{2}m^2)(6m^3n)$.

16. $(7a + 2)(3a + 7)$.

17. If $a = 2$, and $b = 8$, what does $a^2 + 6a^2b$ equal?

18. If $a = 1.3$, and $b = \frac{1}{4}$, what does $\frac{a}{2} + \frac{b}{2}$ equal?

19. If $a = 16$, $b = 3$, $c = .8$, and $d = .16$, what does $a^2 - b + \frac{d}{c}$ equal?

20. If $a = 5$, $b = 3$, and $c = 2$, what does $4\left[\frac{1}{a} + (b - c)^2\right]$ equal?

21. If $d = 2$, $e = 3$, and $f = 4$, what does $\frac{\frac{1}{f} + d}{e^2}$ equal?

22. $d = \frac{e}{f}$. What does e equal?

23. $C = \frac{nE}{R + nr}$. What does R equal?

24. $W = \frac{33000 h}{\pi D n t}$. What does n equal?

25. $2q = 7q + 2$. What does q equal?

26. $21 = 2W - 3$. What does W equal?

27. $2aV + b = c$. What does V equal?

28. $\frac{3u + 2.5}{10} = \frac{7u - 2.5}{15}$. What does u equal?

29. $P = a - g(h_1 - h_2)$. a , g , h_1 , and h_2 are all positive.

Write "larger" or "smaller" or "you cannot know" on the dotted lines.

If a becomes smaller, but not so small as *zero*, P will become

If g becomes smaller, but not so small as *zero*, P will become

If h_1 becomes smaller, but not so small as *zero*, P will become

If h_2 becomes smaller, but not so small as *zero*, P will become

30. $P = a - \frac{1}{b} - c(d - e)^2$. $a = 5$, $b = 4$, $c = 3$, $d = 2$, $e = 1$.

What will P equal:

If a is increased to 6, b , c , d , and e remaining unchanged?

If b is increased to 5, a , c , d , and e remaining unchanged?

If c is increased to 4, a , b , d , and e remaining unchanged?

If d is increased to 3, a , b , c , and e remaining unchanged?

If e is increased to 2, a , b , c , and d remaining unchanged?

31. If the formula for the capacity of any tank in cubic inches is Ubh , what is the formula for its capacity in cubic feet?

32. A certain number plus three fourths of itself equals 7 subtracted from twice the number. What is the number?

Use cross-section paper for Nos. 33, 34, and 35.

33. Make a little cross ($\times$) at the point for which $x = 5$ and $y = 2$.

Make a little circle ($\circ$) at the point for which $x = -3$ and $y = 4$.

Make a little triangle ($\triangle$) at the point for which $x = 2\frac{1}{2}$ and $y = -2$.

34. Draw a graph of $y = 3x + 7$. Mark it No. 34.

35. Draw a graph of $y = \sqrt{x}$. Mark it No. 35. Draw enough of it to show clearly that you understand it.

36. Find two numbers such that:

If twice the second is subtracted from three times the first, the result is 1.

If the second is added to twice the first, the result is 10.

37. Write a formula for a man's total yearly income I , if he has a monthly salary of M dollars, receives a pension of P dollars quarterly, and averages C dollars a week commission.

38. If $a = 2$, $b = 3$, $c = 36$, what does $\sqrt{ab} \sqrt{\frac{1}{c}}$ equal?

39. Write + on the dotted line if the statement is true.

Write - on the dotted line if the statement is false.

$$a^{\frac{1}{3}} = \sqrt[3]{\frac{1}{a}} \quad \text{-----}$$

$$a^{\frac{1}{3}} \times a^{\frac{1}{3}} = \sqrt[3]{a^2} \quad \text{-----}$$

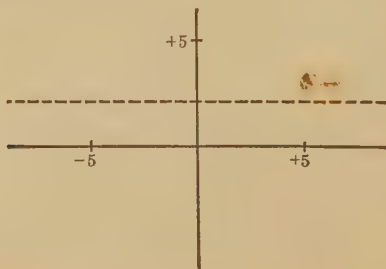
$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}} \quad \text{-----}$$

$$\sqrt{a^5 b} = 2 a \sqrt{b} \quad \text{-----}$$

$$\sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}} \quad \text{-----}$$

40. The dotted line is the graph for $y = 2$.

Draw the graph for $y = 4$.



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$(a+b)^2$ rule page 15
 $(a-b)^2$ " " "

$(a+b)(a-b)$ " " 137

$(a+v)(a+b)$ " " 129

1132
 5/63
 6110
 981
 1261
 10

The square of 13 = 169

" " " 14 = 196

" " " 15 = 225

" " " 16 = 256

" " " 17 = 289

" " " 18 = 324

" " " 19 = 361

" " " 20 = 400

" " " 21 = 441

" " " 22 = 484

#2 - 74 = -10 side

25 - 35 = -10 5 = rule

- 10 - 10

square of side + square of
base = square of hypotenuse

25/25
36/36
49/49

25
36
49

